PERIYAR UNIVERSITY

(NAAC 'A++' Grade with CGPA 3.61 (Cycle - 3)
State University - 25, NIRF Rank 56 - NIRF Innovation Band of 11-50)
SALEM - 636 011

CENTRE FOR DISTANCE AND ONLINE EDUCATION
(CDOE)

MASTER OF COMPUTER APPLICATIONS
SEMESTER - I

CORE – I: DISCRETE MATHEMATICS

(Candidates admitted from 2024 onwards)

Prepared by:
Centre for Distance and Online Education (CDOE)
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Salem – 636011.

Discrete Mathematics

(Theorems and proofs are not expected)

Course Objective

- To know the concepts of relations and functions
- To distinguish among different normal forms and quantifiers
- To solve recurrence relations and permutations & combinations
- To know and solve matrices , rank of matrix & characteristic equations
- To study the graphs and its types

Unit-I

Relations- Binary relations-Operations on relations- properties of binary relations in a set – Equivalence relations–– Representation of a relation by a matrix -Representation of a relation by a digraph – Functions-Definition and examples-Classification of functions-Composition of functions-Inverse function

Unit-II

Mathematical Logic-Logical connectives-Well formed formulas – Truth table of well formed formula –Algebra of proposition –Quine’s method- Normal forms of well formed formulas-Disjunctive normal form-Principal Disjunctive normal form-Conjunctive normal form-Principal conjunctive normal form-Rules of Inference for propositional calculus – Quantifiers- Universal Quantifiers- Existential Quantifiers

Unit-III

Recurrence Relations- Formulation -solving recurrence Relation by Iteration- solving Recurrence Relations- Solving Linear Homogeneous Recurrence Relations of Order Two-Solving Linear Non homogeneous Recurrence Relations. Permutations-Cyclic permutation- Permutations with repetitions- permutations of sets with indistinguishable objects- Combinations- Combinations with repetition

Unit-IV

Matrices- special types of matrices-Determinants-Inverse of a square matrix-Cramer’s rule for solving linear equations-Elementary operations-Rank of a matrix-solving a system of linear equations-characteristic roots and characteristic vectors-Cayley-Hamilton Theorem-problems

Unit-V

Graphs -Connected Graphs -Euler Graphs- Euler line-Hamiltonian circuits and paths – planar graphs – Complete graph-Bipartite graph-Hyper cube graph-Matrix representation of graphs

Text book

1. N.Chandrasekaran and M.Umaparvathi, Discrete mathematics, PHI Learning Private Limited, New Delhi, 2010.

Reference Book

1.Kimmo Eriksson & Hillevi Gavel, Discrete Mathematics & Discrete Models, Studentlitteratur AB, 2015.

2. Kenneth H. Rosen Discrete Mathematics and applications, Mc Graw Hill, 2012

E-content links:

1. **www.cs.uchicago.edu**
2. **www.byjus.com**
3. **www.sciencedirect.com**
4. **www.mathworld.wolfram.com**
5. **www.tutorialspoint.com**
6. **www.reference.wolfram.com**
7. **www.maplesoft.com**
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Unit-I

RELATIONS

Definition :

A binary relation from set A to set B is subset R of AXB

We express particular ordered pair <x,y> € R where x R y

1.1 Operations on relations

1.Union	a (RUS) b if aRb or bRa
2.Intersection	a (R∩S) b if aRb and bRa
3.Difference	a (R-S) b if aRb and b not related a
4.Complementation	a (S-R) b if a not related b and bRa

1.2 Properties of binary relation

Let R be relation on set A. Then

1.R is reflexive if aRa for all a€A

2.R is symmetric if bRa and aRb

3.R is transitive if aRc whenever aRb and bRc

4.R is antisymemetric if bRa and aRb then a=b

5.R is irreflexive if a not related to a

1.3 Equivalence relation

Let R be relation on set A. Then R is equivalence relation if it is

1.Reflexive

2.Symmetric

3.Transitive

1.4 Representation of relation by matrix

Let R be relation on set A to B.

Then M_R is a relation matrix defined by mxn binary matrix given as M_R = m_{ij} Where m_{ij} = 1 if a R b

0 otherwise

1.5 Representation of relation by digraph

Let R be relation on set A =(a1,a2….an) constructs digraph G as follows:

1. The vertices of G are elements of A
2. G has edge connectivity with vertex ai to aj if ai R aj

1.6 Functions

A function is defined as a relation between a set of inputs having one output each.

In simple words, a function is a relationship between inputs where each input is related to exactly one output.

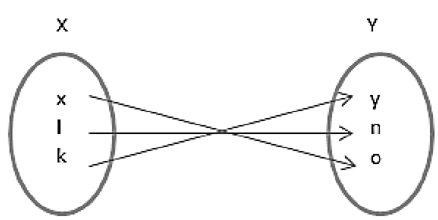
Every function has a domain and codomain or range.

A function is generally denoted by f(x) where x is the input.

Representation

A function f: X →Y is represented as f(x) = y, where, (x, y) ∈ f and x ∈ X and y ∈ Y.

For any function f, the notation f(x) is read as "f of x" and represents the value of y when x is replaced by the number or expression inside the parenthesis. The element y is the image of x under f and x is the pre-image of y under f.

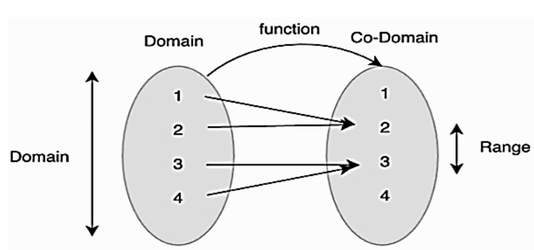


1.7 Range of function

The range of a function refers to all the possible values y could be.

The formula to find the range of a function is y = f(x).

In a relation, it is only a function if every x value corresponds to only one y value

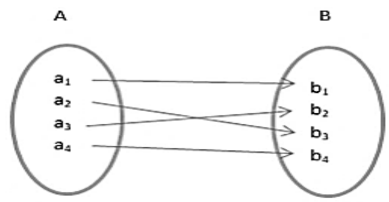


1.8 Classification of functions

One to One Function

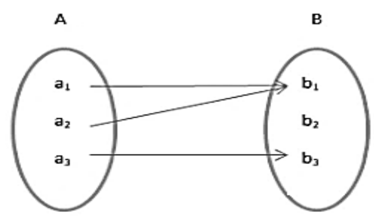
A function f: A → B is One to One if for each element of A there is a distinct element of B. It is also known as Injective.

Consider if a1 ∈ A and a2 ∈ B, f is defined as f: A → B such that f (a1) = f (a2)



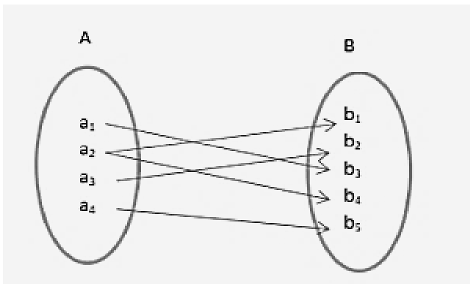
Many to One Function

It is a function which maps two or more elements of A to the same element of set B. Two or more elements of A have the same image in B.



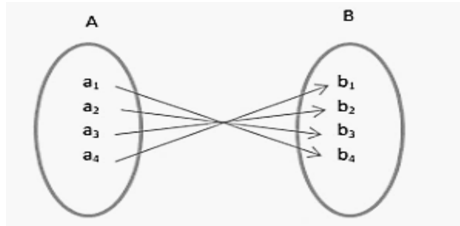
Onto Function

If there exists a function for which every element of set B there is (are) pre-image(s) in set A, it is Onto Function. Onto is also referred as Surjective Function.



One – One and Onto Function

A function, f is One – One and Onto or Bijective if the function f is both One to One and Onto function. In other words, the function f associates each element of A with a distinct element of B and every element of B has a pre-image in A.



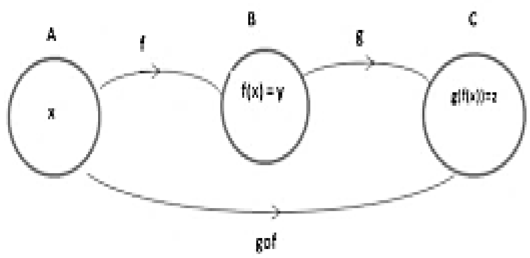
1.9 Composition of function

The composition of a function is a step-wise application.

For example, the function f: A→ B & g: B→ C can be composed to form a function which maps x in A to g(f(x)) in C.

All sets are non-empty sets. A composite function is denoted by (g o f) (x) = g (f(x)).

The notation g o f is read as "g of f".

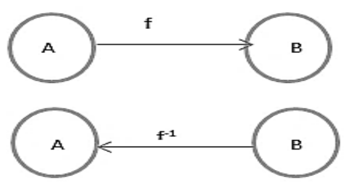


1.10 Invertible Function

A function is invertible if on reversing the order of mapping we get the input as the new output.

In other words, if a function, f whose domain is in set A and image in set B is invertible if f-1 has its domain in B and image in A.

f(x) = y ⇔ f-1 (y) = x.



Examples

1. If f: A → B, f(x) = y = x2 and g: B→C, g(y) = z = y + 2 find g o f.

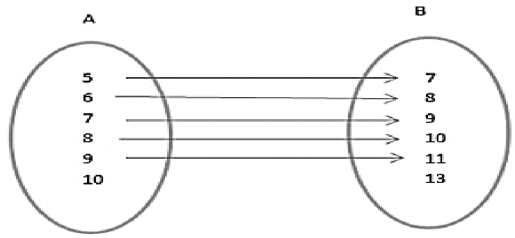
Given A = {1, 2, 3, 4, 5}, B = {1, 4, 9, 16, 25}, C = {2, 6, 11, 18, 27}.

Answer : gof(x) = g(f(x))

g(f(1)) = g(1) = 2, g(f(2)) = g(4) = 6, g(f(3)) = g(9) = 11, g(f(4)) = g(16) = 18, g(f(5)) = g(25) = 27.

2. Let A = {5, 6, 7, 8, 9, 10} and B = {7, 8, 9, 10, 11, 13}. Define a relation R from A to B by R = {(x, y): y = x + 2}. Write down the domain, codomain and range of R.

Answer : Here, R = {(5, 7), (6, 8), (7, 9), (8, 10), (9, 11)}.



Domain = {5, 6, 7, 8, 9}

Range = {7, 8, 9, 10, 11}

Co-domain = {7, 8, 9, 10, 11, 13}.

3. Three friends A, B, and C live near each other at a distance of 5 km from one another. We define a relation R between the distances of their houses. Is R an equivalence relation?

Solution:

For an equivalence Relation, R must be reflexive, symmetric and transitive.

R is not reflexive as A cannot be 5 km away to itself.

The relation, R is symmetric as the distance between A & B is 5 km which is the same as the distance between B & A.

R is transitive as the distance between A & B is 5 km, the distance between B & C is 5 km and the distance between A & C is also 5 km.

Therefore, this relation is not equivalent.

4. Suppose set A = {1,2,3,4} and Set B = {0,2,4,6} and relation aRb such that a < b. Using the roster method, list the elements of R.

Solution:

Given A = {1,2,3,4} and B = {0,2,4,6}

R={(a,b) € AXB(a < b)}

Therefore R = {(1,2)(1,4)(1,6)(2.4)(2,8)(3,4)(3,6)(4,6)}

5. Assume set A = {0,1,2} and Set B = {0,1,2,3} and let relations R = {(0,0),(1,1), (2,2))} and T = {(0,0),(0,1),(0,2),(0,3)}.

Let $A = \{0,1,2\}$ and $B = \{0,1,2,3\}$ with relations $R = \{(0,0),(1,1),(2,2)\}$ and $T = \{(0,0),(0,1),(0,2),(0,3)\}$

$$R \cup T = \{(0,0),(0,1),(0,2),(0,3),(1,1),(2,2)\}$$

$$R \cap T = \{(0,0)\}$$

$$R - T = \{(1,1),(2,2)\}$$

$$T - R = \{(0,1),(0,2),(0,3)\}$$

$$R \oplus T = \{(0,1),(0,2),(0,3),(1,1),(2,2)\}$$

6.suppose we let A = {1,2,3} , B = {0,1,2}, and C = {a,b}. If T = {(0,b),(1,a),(2,b)} is a relation from B to C and R = {(1,0),(2,2),(3,1),(3,2)} is a relation from A to B, the the composite of the relations T and R is found by matching the range (2nd elements) of R with the domain (1st elements) of

$$R \circ T = \{(1,0),(2,2),(3,1),(3,2)\} \circ \{(0,b),(1,a),(2,b)\}$$

$$R \circ T = \{(1,b),(2,b),(3,a),(3,b)\}$$

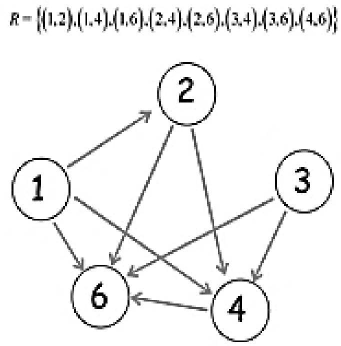
7. Using a previous example where set A = {1,2,3,4} and set B = {0,2,4,6} with relation aRb such that a < b, we found that R = {(1,2),(1,4),(1,6),(2,4),(2,6),(3,4),(3,6),(4,6)}. This relation can be displayed as an incidence matrix shown below

$A = \{1,2,3,4\}$, $B = \{0,2,4,6\}$ and $R = \{(a,b) \in A \times B \mid a < b\}$, $|A| = 4$ and $|B| = 4$ then $M_R = 4 \times 4$

$$R = \{(1,2),(1,4),(1,6),(2,4),(2,6),(3,4),(3,6),(4,6)\}$$

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} 0 & 2 & 4 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8. we can display relation R = {(1,2),(1,4),(1,6),(2,4),(2,6),(3,4),(3,6),(4,6)} as a directed graph, where each element in R becomes a vertex, and a directed edge connects these vertices if they are an ordered pair in the relation, as shown below.



9. Find the output of the function g(t) = 6t2 + 5 at

(i) t = 0

(ii) t = 2

Solution:

The given function is g(t) = 6t2 + 5

(i) At t = 0, g(0) = 6(0)2 + 5 = 5

(ii) At t = 2, g(2) = 6(2)2 + 5 = 29

10. For the given functions f(x) = 3x + 2 and g(x) = 2x - 1, find the value of fog(x).

Solution:

The given two functions are f(x) = 3x + 2 and g(x) = 2x - 1.

We need to find the function fog(x).

fog(x) = f(g(x))

= f(2x-1)

= 3(2x - 1) + 2

= 6x - 3 + 2

= 6x - 1

Answer: Therefore fog(x) = 6x – 1

11. Find the inverse function of the function f(x) = 5x + 4.

Solution:

The given function is f(x) = 5x + 4

we rewrite it as y = 5x + 4 and simplify it to find the value of x.

y = 5x + 4

y - 4 = 5x

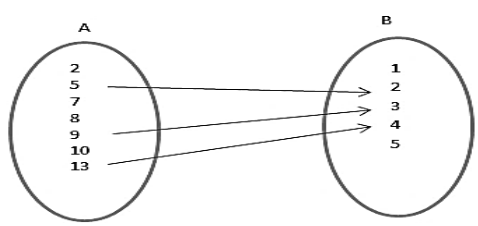
x = (y - 4)/5

$f^{-1}(x) = (x - 4)/5$

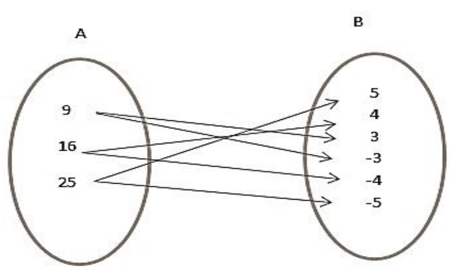
Answer**:** Therefore the inverse function is $f^{-1}(x) = (x - 4)/5$

12. Which of the following is a function?

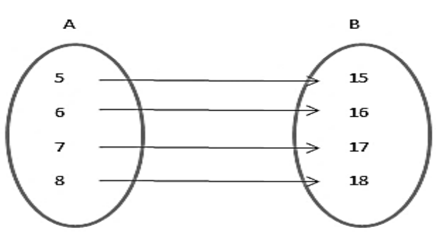
1 .



2.



3.



Answer :

Figure 3 is an example of function since every element of A is mapped to a unique element of B and no two distinct elements of B have the same pre-image in A.

13. Give an example of a function?

Answer: An example of a function is the relationship x → x. The reason for this is that every element in x has a relation with y. Moreover, no element in x has two or more than two relationships.

14. Explain what is a function and what is not?

Answer: A function refers to a relation such that every input has only one output. For example, y is a function of x and x is not a function of y (y = 9 consist of multiple outputs). Moreover, y is not a function of x (x = 1 consist of multiple outputs), x does not happen to be a function of y (y = 2 has multiple outputs).

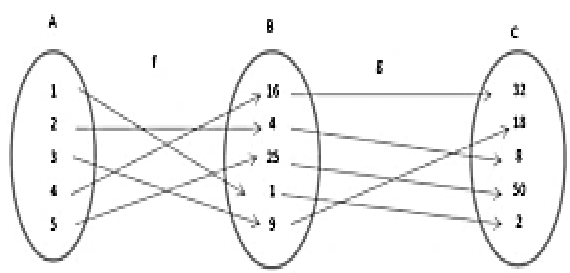
15. What is the classification of functions?

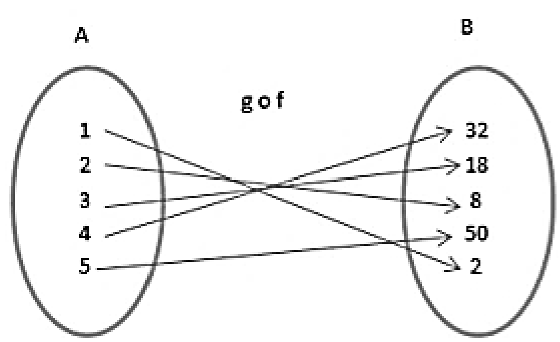
Answer: The classification of function takes place by the type of mathematical equation which shows their relationship. Some functions can be algebraic. Trigonometric functions like f(x) = sin x are those that involve angles. Some functions have logarithmic and exponential and logarithmic relationships and their classification are as such.

16. What are basic polynomial functions?

Answer: (x) =c, f(x) =x, f(x) =x2, and f(x) =x3 are basic polynomial functions.

17. Consider the functions f: A→B and g: B→C. f = {1, 2, 3, 4, 5}→ {1, 4, 9, 16, 25} and g = {1, 4, 9, 16, 25} → {2, 8, 18, 32, 50}. A = {1, 2, 3, 4, 5}, B = {16, 4, 25, 1, 9}, C = {32, 18, 8, 50, 2}.Here, g o f = {(1, 2), (2, 8), (3, 18), (4, 32), (5, 50)}.





18. Let us assume that F is a relation on the set R real numbers defined by xFy if and only if x-y is an integer. Prove that F is an equivalence relation on R.

Solution:

Reflexive: Consider x belongs to R,then x – x = 0 which is an integer. Therefore xFx.

Symmetric: Consider x and y belongs to R and xFy. Then x – y is an integer.

Thus, y – x = – (x – y), y – x is also an integer. Therefore yFx.

Transitive: Consider x and y belongs to R, xFy and yFz. Therefore x-y and y-z are integers. According to the transitive property, (x – y) + (y – z) = x – z is also an integer. So that xFz.

Thus, R is an equivalence relation on R.

19. Show that the relation R is an equivalence relation in the set A = { 1, 2, 3, 4, 5 } given by the relation R = { (a, b):|a-b| is even }.

Solution:

R = { (a, b):|a-b| is even }. Where a, b belongs to A

Reflexive Property :

From the given relation,

|a – a| = | 0 |=0

And 0 is always even.

Thus, |a-a| is even

Therefore, (a, a) belongs to R

Hence R is Reflexive

Symmetric Property :

From the given relation,

|a – b| = |b – a|

We know that |a – b| = |-(b – a)|= |b – a|

Hence |a – b| is even,

Then |b – a| is also even.

Therefore, if (a, b) ∈ R, then (b, a) belongs to R

Hence R is symmetric.

Transitive Property :

If |a-b| is even, then (a-b) is even.

Similarly, if |b-c| is even, then (b-c) is also even.

Sum of even number is also even

So, we can write it as a-b+ b-c is even

Then, a – c is also even

So,

|a – b| and |b – c| is even , then |a-c| is even.

Therefore, if (a, b) ∈ R and (b, c) ∈ R, then (a, c) also belongs to R

Hence R is transitive.

Self-assessment questions

1. Write the operations on relations with example
2. Explain the properties of binary relation
3. Elucidate equivalence relation
4. Explain representation of relation by digraph with example
5. Analyze the range of function
6. Elaborate Classification of functions
7. Explain composition of function with example
8. Summarize Invertible Function

Let us sum up

Binary Relations: Describe a relationship between elements of two sets.

Functions: A special kind of binary relation where each element of the domain maps to exactly one element of the codomain.

Check your progress

1. A relation R in a set A is called _______, if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

(a) symmetric

(b) transitive

(c) equivalence

(d) non-symmetric

2. Let R be a relation on the set N of natural numbers defined by nRm if n divides m. Then R is

(a) Reflexive and symmetric

(b) Transitive and symmetric

(c) Equivalence

(d) Reflexive, transitive but not symmetric

3. The maximum number of equivalence relations on the set A = {1, 2, 3} are

(a) 1

(b) 2

(c) 3

(d) 5

4. If set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is

(a) 720

(b) 120

(c) 0

(d) none of these

5. Let $f : [2, \infty) \rightarrow R$ be the function defined by $f(x) = x^2 – 4x + 5$, then the range of f is

(a) R

(b) $[1, \infty)$

(c) $[4, \infty)$

(d) $[5, \infty)$

Correct option: (b) $[1, \infty)$

6. Let $f : R \rightarrow R$ be defined by $f(x) = 1/x \ \forall\ x \in R$. Then f is

(a) one-one

(b) onto

(c) bijective

(d) f is not defined

7. Let A = {1, 2, 3} and consider the relation R = {1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1,3)}. Then R is

(a) reflexive but not symmetric

(b) reflexive but not transitive

(c) symmetric and transitive

(d) neither symmetric, nor transitive

8. If f : R → R be defined by $f(x) = 3x^2 – 5$ and g : R → R by $g(x) = x/(x^2 + 1)$, then g o f is

(a) $(3x^2 – 5)/(9x^4 – 30x^2 + 26)$

(b) $(3x^2 – 5)/(9x^4 – 6x^2 + 26)$

(c) $3x^2/(x^4 + 2x^2 – 4)$

(d) $3x^2/(9x^4 + 30x^2 – 2)$

9. Let f : R → R be given by f (x) = tan x. Then $f^{-1}(1)$ is

(a) π/4

(b) {n π + π/4 : n ∈ Z}

(c) does not exist

(d) none of these

10. If f: R → R be given by $f(x) = (3 – x^3)^{1/3}$, then fof(x) is

(a) $x^{1/3}$

(b) x^3

(c) x

(d) $(3 – x^3)$

11. Let f : A → B and g : B → C be the bijective functions. Then $(g \circ f)^{-1}$ is

(a) $f^{-1} \circ g^{-1}$

(b) $f \circ g$

(c) $g^{-1} \circ f^{-1}$

(d) $g \circ f$

Unit Summary

Binary relations and functions are key concepts in mathematics, particularly in set theory and logic. A **binary relation** between two sets A and B is a subset of the Cartesian product A×Btimes

A×B, consisting of ordered pairs (a,b) where a∈Aa and b∈Bb . The **domain** of a relation is the set of all first elements in the pairs, while the **range** is the set of all second elements. Binary relations can exhibit several important properties. A relation is **reflexive** if every element in the domain is related to itself, meaning that for all a∈Aa , the pair (a,a) must be included in the relation. It is **symmetric** if for every pair (a,b) in the relation, the reverse pair (b,a) is also present. A relation is **transitive** if whenever (a,b)and (b,c) are in the relation, then (a,c) must also be in the relation. Finally, a relation is **antisymmetric** if for any (a,b) and (b,a) in the relation, a must equal b.

A **function** from set A to set B is a special type of relation where each element of A is associated with exactly one element of B. The **domain** of a function is the set A from which inputs are drawn, while the **codomain** is the set B of potential outputs. The **range** of a function is the subset of the codomain consisting of all actual outputs produced by the function. Functions can be classified based on their properties. A function is **injective** or one-to-one if different elements in the domain map to distinct elements in the codomain. If every element in the codomain is the image of at least one element in the domain, the function is **surjective** or onto. A function is **bijective** if it is both injective and surjective, meaning there is a one-to-one correspondence between the elements of the domain and codomain.

Suggested readings

1. Discrete Mathematics and Its Applications" by Kenneth H. Rosen
2. Discrete Mathematics with Applications" by Susanna S. Epp

Glossary

1. Binary Relation

A binary relation R from a set A to a set B is a subset of the Cartesian product A×Btimes A×B.

2. Domain

The domain of a binary relation R is the set of all first elements of the ordered pairs in R. It can be denoted as dom(R)

3. Range

The range of a binary relation R is the set of all second elements of the ordered pairs in R

4. Codomain

The codomain of a binary relation R is the set B in the context of a relation from set A to set B .It represents the set of all possible second elements in the relation.

5. Function Notation

Function notation is the way of representing functions using symbols. For example, f(x) denotes the output of the function f when the input is x.

6. Multivalued Function

A multivalued function assigns multiple outputs to a single input. Unlike standard functions, which map each input to exactly one output, multivalued functions are more general and are often used in advanced mathematics.

7. Real-Valued Function
A real-valued function is a function where the codomain is the set of real numbers R. It maps elements from the domain to real numbers.
8. Mathematical Function
A mathematical function is a rule or mapping that assigns exactly one output value to each input value within its domain. It can be expressed as an equation, graph, or algorithm.

Unit-II

2.1 Mathematical logic

2.2 Logical connectives

2.3 Well Formed Formulas (WFF)

2.4 Statement Formulas

2.5 Rules of the Well-Formed Formulas

2.6 Example Of Well Formed Formulas

2.7 Truth table for WFF

2.8 Algebra of propositions

2.9 Quine’s method

2.10 Normal forms

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2.12 Quantifiers

Unit-II

2.1 Mathematical logic

In mathematical logic, a term denotes a mathematical object while a formula denotes a mathematical fact. In particular, terms appear as components of a formula.

This is analogous to natural language, where a noun phrase refers to an object and a whole sentence refers to a fact.

2.2 Logical connectives

A Logical Connective is a symbol which is used to connect two or more propositional or predicate logics in such a manner that resultant logic depends only on the input logics and the meaning of the connective used.

Generally there are five connectives which are −

OR (∨)

AND (∧)

Negation/ NOT (¬)

Implication / if-then (→)

If and only if (⇔).

OR (∨) − The OR operation of two propositions A and B (written as A ∨ B) is true if at least any of the propositional variable A or B is true.

The truth table is as follows –

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

AND (∧) − The AND operation of two propositions A and B (written as A∧B) is true if both the propositional variable A and B is true.

The truth table is as follows –

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Negation (¬) − The negation of a proposition A (written as ¬ A) is false when A is true and is true when A is false.

The truth table is as follows −

P	$\neg P$
T	F
F	T

Implication / if-then (→) − An implication A → B is the proposition "if A, then B". It is false if A is true and B is false. The rest cases are true.

The truth table is as follows −

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

If and only if (⇔) − A ⇔ B is bi-conditional logical connective which is true when p and q are same, i.e. both are false or both are true.

The truth table is as follows −

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

2.3 Well Formed Formulas (WFF)

Well-Formed Formula(WFF) is an expression consisting of variables(capital letters),parentheses, and connective symbols. An expression is basically a combination of operands & operators and here operands and operators are the connective symbols.

Below are the possible Connective Symbols:

¬ (Negation)

∧ (Conjunction)

∨ (Disjunction)

⇒ (Rightwards Arrow)

⇔ (Left-Right Arrow)

2.4 Statement Formulas

1. Statements that do not contain any connectives are called Atomic or Simple statements and these statements in themselves are WFFs.

For example,

P, Q, R, etc.

2. Statements that contain one or more primary statements are called Molecular or Composite statements

For example,

If P and Q are two simple statements, then some of the Composite statements which follow WFF standards can be formed are:

-> ¬P

-> ¬Q

-> (P ∨ Q)

-> (P ∧ Q)

-> (¬P ∨ Q)

-> ((P ∨ Q) ∧ Q)

-> (P ⇒ Q)

-> (P ⇔ Q)

-> ¬(P ∨ Q)

-> ¬(¬P ∨ ¬Q)

2.5 Rules of the Well-Formed Formulas

1. A Statement variable standing alone is a **Well-Formed Formula(WFF)**.
   For example– Statements like P, ~P, Q, ~Q are themselves Well Formed Formulas.
2. If 'P' is a WFF then ~P is a formula as well.
3. If P & Q are WFFs, then (P∨Q), (P∧Q), (P⇒Q), (P⇔Q), etc. are also WFFs.

2.6 Example Of Well Formed Formulas:

WFF	**Explanation**
¬¬P	By **Rule 1** each Statement by itself is a WFF, ¬P is a WFF, and let ¬P = Q. So ¬Q will also be a WFF.
((P⇒Q)⇒Q)	By **Rule 3** joining '(P⇒Q)' and 'Q' with connective symbol '⇒'.
(¬Q ∧ P)	By **Rule 3** joining '¬Q' and 'P' with connective symbol '∧'.
((¬P∨Q) ∧ ¬¬Q)	By **Rule 3** joining '(¬P∨Q)' and '¬¬Q' with connective symbol '∧'.
¬((¬P∨Q) ∧ ¬¬Q)	By **Rule 3** joining '(¬P∨Q)' and '¬¬Q' with connective symbol '∧' and then using Rule 2.

Below are the Examples which may seem like a WFF but they are not considered as Well-Formed Formulas:

1. **(P)**, 'P' itself alone is considered as a WFF by Rule 1 but placing that inside parenthesis is not considered as a WFF by any rule.
2. **¬P ∧ Q**, this can be either (¬P∧Q) or ¬(P∧Q) so we have ambiguity in this statement and hence it will not be considered as a WFF. Parentheses are mandatory to be included in Composite Statements.

3. **((P ⇒ Q))**, We can say (P⇒Q) is a WFF and let (P⇒Q) = A, now considering the outer parentheses, we will be left with (A), which is not a valid WFF. Parentheses play a really important role in these types of questions.
4. **(P ⇒⇒ Q)**, connective symbol right after a connective symbol is not considered to be valid for a WFF.
5. **((P ∧ Q) ∧)Q)**, conjunction operator after (P∧Q) is not valid.
6. **((P ∧ Q) ∧ PQ)**, invalid placement of variables(PQ).
7. **(P ∨ Q) ⇒ (∧ Q)**, with the Conjunction component, only one variable 'Q' is present. In order to form an operation inside a parentheses minimum of 2 variables are required.

2.7 Truth table for WFF

Constructing Truth Tables for Compound Wffs

Examples

1. Construct a truth table for the wff:

$(A' \to B) \wedge (C' \vee A)$

Solution:

A	B	C	A'	$A' \to B$	C'	$C' \vee A$	$(A' \to B) \wedge (C' \vee A)$
T	T	T	F	T	F	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	F	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	F	F	F
F	T	F	T	T	T	T	T
F	F	T	T	F	F	F	F
F	F	F	T	F	T	T	F

2. Construct a truth table for the formula

$$\neg P \wedge (P \to Q)$$

Solution:

P	Q	$\neg P$	$P \to Q$	$\neg P \wedge (P \to Q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Note:

Two statements X and Y are **logically equivalent** if $X \leftrightarrow Y$ is a tautology.

3. Show that $P \to Q$ and $\neg P \vee Q$ are logically equivalent.

Solution:

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the columns for $P \to Q$ and $\neg P \vee Q$ are identical, the two statements are logically equivalent

4. Construct a truth table for $(P \to Q) \wedge (Q \to R)$

Solution:

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \wedge (Q \to R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Note:

1.A tautology is a formula which is "always true" --- that is, it is true for every assignment of truth values to its simple components. You can think of a tautology as a rule of logic.

2.The opposite of a tautology is a contradiction, a formula which is "always false". In other words, a contradiction is false for every assignment of truth values to its simple components.

Example:

5.Show that $(P \to Q) \vee (Q \to P)$ is a tautology.

Solution:

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

2.7 Algebra of propositions

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Logical Equivalences	Set Properties
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$ b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge \sim p \equiv \mathbf{c}$	a. $A \cup A^c = U$ b. $A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t} \equiv \mathbf{t}$ b. $p \wedge \mathbf{c} \equiv \mathbf{c}$	a. $A \cup U = U$ b. $A \cap \emptyset = \emptyset$
a. $\sim(p \vee q) \equiv \sim p \wedge \sim q$ b. $\sim(p \wedge q) \equiv \sim p \vee \sim q$	a. $(A \cup B)^c = A^c \cap B^c$ b. $(A \cap B)^c = A^c \cup B^c$
a. $p \vee (p \wedge q) \equiv p$ b. $p \wedge (p \vee q) \equiv p$	a. $A \cup (A \cap B) \equiv A$ b. $A \cap (A \cup B) \equiv A$
a. $\sim \mathbf{t} \equiv \mathbf{c}$ b. $\sim \mathbf{c} \equiv \mathbf{t}$	a. $U^c = \emptyset$ b. $\emptyset^c = U$

Table 6.4.1

2.9 Quine's method

Let's remember the basics of Quine's method. Given a proposition p, we simplify p, i.e., we produce a new proposition p' which logically equivalent to p, such that p' is either:

- Boolean True, in which case it is a tautology,
- Boolean False, in which case it is refutable, i.e., not a tautology, or
- there are no Boolean nodes remaining, in which case we
    1. select a variable v which occurs in p.
    2. create two new propositions pt and pf, by substituting Boolean True and Boolean False for v in the original proposition. Then p is a tautology if and only if both pt and pf are.

2.10 NORMAL FORMS

Principal conjunctive and disjunctive normal forms

Normal forms

By constructing and comparing truth tables we can determine whether two statement formulas A and B are equivalent. But this method is very tedious and difficult to solve even on a computer because the number of entries increases very rapidly as n increases. A better method is to transform the statement formulas A and B to some standard forms A's and B' such that a simple comparison of A and B shows whether A ⇔ B.

The standard forms are called canonical forms or normal forms.

Let A (P_1, P_2, … P_n) be a statement formula where P_1, P_2, ... P_n are the primitive variables. If A has the truth value T for atleast one combination of truth values assigned to P_1, P_2, … P_n then A is said to be satisfiable..

Note: It will be convenient to use the word "product" in place of "conjunction" and "sum" in place of "disjunction".

Definition: Elementary product

A product of the variables and their negations in a formula is called an elementary product. (product means conjunction).

Example: Let P and Q be any two atomic variables. Then P, ┐P^Q, ┐Q^P, P^┐P and Q^┐P are elementary products.

Definition: Elementary sum

A sum of the variables and their negations in a formula is called an elementary sum. (Sum means disjunction).

Example: Let P and Q be any two variables. Then P, ┐PVQ, ┐QVP, PV┐P and QV┐P are elementary sums.

Definition: Factor

Any part of an elementary product or elementary sum, which is itself an elementary product of sum is a factor of the product or sum.

Example: Q v P is a factor of ┐QvQVP

Note 1: An elementary product is identically false if it contains atleast one pair of factors in which one is negation of the other.

Example: P^┐P ↔ F

Note 2: An elementary sum is identically true if it contains atleast one pair of factors in which one is negation of the other.

Example: $P \vee \neg P \leftrightarrow T$.

Disjunctive Normal Form (DNF)

Definition

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form (DNF) of the given formula.

Procedure to obtain DNF

1. An equivalent formula can be obtained by replacing $\rightarrow$ and $\leftrightarrow$ with ^, V and $\neg$.

2. Apply negation to the formula or to a part of the formula and not to the variables.

3. Using DeMorgan's law, apply negation to variables.

4. Repeated application of distributive laws will give the required DNF.

Remark :

1. A given formula may not have unique DNF. We may get different DNF's if we apply distributive law in different ways.

2. A given formula is false if every elementary product in DNF is identically false

Example 6**:** Obtain disjunctive normal forms of $P \wedge (P \rightarrow Q)$"

Solution Let $S \leftrightarrow P \wedge (P \rightarrow Q)$

$\leftrightarrow P \wedge (\neg P \vee Q)$ $[P \rightarrow Q \leftrightarrow \neg P \vee Q]$

$\leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

[by distributive law]

Example 7**:** Obtain disjunctive normal forms of $\neg (P \vee Q) \leftrightarrow P \wedge Q$.

Solution: Let $S \leftrightarrow \neg (P \vee Q) \leftrightarrow P \wedge Q$

$\leftrightarrow [\neg (P \vee Q) \wedge (P \wedge Q)] \vee [\neg \neg (P \vee Q) \wedge (\neg P \wedge Q)]$

$[R \leftrightarrow S \leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)\}$

$\leftrightarrow [\neg (P \vee Q) \wedge (P \wedge Q)] \vee [(P \vee Q) \wedge \neg (P \wedge Q)]$ [by negation law]

$\leftrightarrow [(\neg P \wedge \neg Q) \wedge (P \wedge Q)] \vee [(P \vee Q) \wedge (\neg P \vee \neg Q)]$ [by DeMorgan's law]

$\leftrightarrow [\neg P \wedge \neg Q) \wedge (P \wedge Q)] \vee [[(P \vee Q) \wedge \neg P] \vee [(P \vee Q) \wedge \neg Q)]$ ['. by distributive law]

$\leftrightarrow [(\neg P \wedge 7Q) \wedge (P \wedge Q)] \vee [[(P \wedge \neg P) \vee Q \wedge \neg P)] \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)]]$ [by distributive law]

$\leftrightarrow [(\neg P \wedge \neg Q) \wedge (P \wedge Q)] \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$

Example 8**:** Obtain a disjunctive normal form of $P \rightarrow ((P \rightarrow Q) \wedge \neg (\neg Q \vee \neg P))$

Solution :

Let $\$ \Leftrightarrow P \rightarrow ((P \rightarrow Q) \wedge \neg (\neg Q \vee \neg P))$

$\leftrightarrow \neg P \vee ((P \rightarrow Q) \wedge \neg (\neg Q \vee \neg P))$

['.' $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$]

$\leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge (Q \wedge P)]$

['.' $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ and Demorgan's law]

$\leftrightarrow \neg P \vee [(\neg P \wedge (Q \wedge P)) \vee (Q \wedge (Q \wedge P))]$ ['.' Distributive law]

$\leftrightarrow \neg P \vee (\neg P \wedge (Q \wedge P)] \vee [Q \wedge (Q \wedge P)]$

$\leftrightarrow \neg P \vee [\neg P \wedge (Q \wedge P)] \vee [Q \wedge Q) \wedge P]$ ['.' Associative law]

$\leftrightarrow \neg P \vee [\neg P \wedge (Q \wedge P)] \vee [Q \wedge P]$

['.' $Q \wedge Q \Leftrightarrow Q$]

Example 9**:** Obtain a disjunctive normal form of

$P \wedge \neg (Q \vee R)) \vee (((P \wedge Q) \vee \neg R) \wedge P)$

Solution: $P \wedge \neg (Q \vee R)) \vee (((P \wedge Q) \vee \neg R) \wedge P)$ Reasons

$\leftrightarrow (P \wedge (\neg Q \wedge \neg R)) \vee (((P \wedge Q) \vee \neg R) \wedge P)$ De Morgan's law

$\leftrightarrow (P \wedge \neg Q \wedge \neg R)) \vee ((P \wedge Q) \wedge P) \vee (\neg R \wedge P)$ Distributive law

$\leftrightarrow (P \wedge \neg Q \wedge \neg R) \vee ((P \wedge (Q \wedge P)) \vee (\neg R \wedge P))$ Associative law

$\leftrightarrow (P \wedge \neg Q \wedge \neg R) \vee ((P \wedge (P \wedge Q)) \vee (\neg R \wedge P))$ Commutative law

$\leftrightarrow (P \wedge \neg Q \wedge \neg R) \vee ((P \wedge P) \wedge Q)) \vee (\neg R \wedge P))$ Associative law

$\leftrightarrow (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q) \vee (\neg R \wedge P)$ Idempotent laws

This is the required DNF, as it is a sum of elementary products.

Example 10**:** Obtain a disjunctive normal form of $(Q \vee (P \wedge R)) \wedge ((P \vee R) \wedge Q)$

Solution: $(Q \vee (P \wedge R)) \wedge ((\neg P \vee R) \wedge 'Q)$ Reason

$\leftrightarrow (Q \vee (P \wedge R)) \wedge ((\neg P \vee R) \vee \neg Q)$ De Morgan law

$\leftrightarrow (Q \vee (P \wedge R)) \wedge ((\neg P \wedge \neg R) \vee \neg Q)$ De Morgan law

$\leftrightarrow (Q \wedge (\neg P \wedge \neg R)) \vee (Q \wedge \neg Q) \vee ((P \wedge R) \wedge \neg P \wedge \neg R) \vee ((P \wedge R) \wedge \neg Q)$ extended distributive law

$\leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee F \vee (F \wedge R) \vee (P \wedge \neg Q \wedge R)$ Negation law

$\leftrightarrow (\neg p \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$ Domination law

This is the required DNF, as it is a sum of elementary products.

Example 11**:** Obtain the DNF for $(P\rightarrow(Q\wedge R))\wedge((\neg P\rightarrow 7\ Q)\wedge\neg R))$

Solution :

$P\rightarrow(Q\wedge R))\wedge((\neg P\rightarrow\neg Q)\wedge\neg R)$ Reasons

$\leftrightarrow (\neg P \vee(Q\wedge R))\wedge((P \vee \neg Q) \wedge \neg R)$ De Morgan and Double negation

$\leftrightarrow (\neg P \vee(Q\wedge R))\wedge((P \wedge \neg R) \vee(\neg Q \wedge \neg R)$ Distributive law

$\leftrightarrow (\neg P \wedge (P \wedge \neg R)) \vee (\neg P \wedge (\neg Q \wedge \neg R)) \vee ((Q\wedge R) \wedge (P \wedge \neg R)) \vee ((Q\wedge R) \wedge (\neg Q \wedge \neg R))$ extended distributive law

$((\neg P\wedge P) \wedge \neg R) \vee (\neg P\wedge (\neg Q \wedge \neg R)) \vee ((Q \wedge R) \wedge (P \wedge \neg R)) \vee ((Q\wedge R) \wedge (\neg Q\wedge\neg R))$ Associative law.

$\leftrightarrow (F \wedge \neg R) \vee (\neg P \wedge (\neg Q \wedge \neg R)) \vee ((Q \wedge R) \wedge (P \wedge \neg R)) \vee ((Q\wedge R) \wedge (\neg Q\wedge\neg R))$ Negation law

$\leftrightarrow (F \vee (\neg P \wedge (\neg Q \wedge \neg R)) \vee ((Q \wedge R) \wedge (P \wedge \neg R)) \vee ((Q \wedge R) \wedge (\neg Q \wedge \neg R))$ Domination law

$\leftrightarrow (\neg P \wedge (\neg Q \wedge \neg R)) \vee ((Q \wedge R) \wedge (P \wedge \neg R)) \vee ((Q \wedge R) \wedge (\neg Q \wedge \neg R))$ Identity law

This is the required DNF, as it is a sum of elementary products.

Example 12**:** Obtain the disjunctive normal form of

$(\neg P \vee \neg Q) \rightarrow (\neg P \wedge R)$.

Solution: $(\neg P \vee \neg Q) \rightarrow (\neg P \wedge R)$ Reasons

$\leftrightarrow (\neg P \vee \neg Q) \vee (\neg P \wedge R)$ since $P\rightarrow Q \leftrightarrow \neg P \vee Q$

$\leftrightarrow (P \wedge Q) \vee (\neg P \wedge R)$ De Morgan law

which is the required DNF

Example 13: Obtain the disjunctive normal form of

$(\neg P\rightarrow \neg Q) \vee (P\downarrow Q)$

Solution $(\neg P\rightarrow \neg Q) \vee (P\downarrow Q)$ Reasons

$\leftrightarrow (P \vee \neg Q) \vee (P \vee Q)$ $(P \rightarrow Q) \leftrightarrow \neg P \vee Q$

$(P\downarrow Q) \leftrightarrow \neg (P \vee Q)$

$\leftrightarrow \neg (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$ De Morgan law

which is the required DNF.

Conjunctive Normal Form :

A formula. which is equivalent to a given formula and which consists of a product of elementary sums is called a conjunctive normal form of the given formula.

Remarks:

(1) Again CNF is not unique for a formula.

(2) A given formula is identically true if every elementary sum in CNF is identically true.

Example 14**:** Obtain a conjunctive normal form of $P \wedge (P \rightarrow Q)$.

Solution : $P \wedge (P \rightarrow Q)$ Reason

$\leftrightarrow P \wedge (\neg P \vee Q)$

which is in CNF as P, $\neg P \vee Q$ are elementary sums.

Example 15**:** Obtain a conjunctive normal form of $\neg (P \vee Q) \leftrightarrow (P \wedge Q)$.

Solution: $\neg (P \vee Q) \leftrightarrow (P \wedge Q)$ Reasons

$\leftrightarrow (\neg (P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \neg (P \vee Q))$ $\quad R \leftrightarrow S \leftrightarrow (R \rightarrow S) \wedge (S \rightarrow R)$

$\leftrightarrow (P \vee Q) \vee (P \wedge Q) \wedge \neg (P \wedge Q) \vee \neg (P \vee Q)$ $\quad R \rightarrow S \leftrightarrow \neg R \vee S$

$\leftrightarrow P \vee Q \vee (P \wedge Q) \wedge (\neg P \vee \neg Q \vee (\neg P \wedge \neg Q))$ De Morgan's law

$\leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$ using distributive law

This is required CNF, as each of $P \vee Q \vee P$, $\neg P \vee \neg Q \vee \neg P$, $\neg P \vee \neg Q \vee \neg Q$ is an elementary sum.

Example 16**:** Obtain a CNF for $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

Solution :

$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ Reasons

$\leftrightarrow (Q \vee ((P \vee \neg P) \quad \neg Q)$ $\quad (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \leftrightarrow (P \vee \neg P) \wedge \neg Q)$ distributive law

$\leftrightarrow (Q \vee (P \vee \neg P)) \wedge (Q \vee \neg Q)$ distributive law

$\leftrightarrow (Q \vee (P \vee \neg P) \wedge (Q \vee \neg Q)$

This is a CNF, as $Q \vee P \vee \neg P$, $Q \vee \neg Q$ are elementary sums.

Example 17**:** Obtain a CNF for $(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow \neg Q \wedge \neg R))$

Solution : $(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow \neg Q \wedge \neg R))$ Reasons

$\leftrightarrow (\neg P \vee (Q \wedge R)) \wedge (P \vee (\neg Q \wedge \neg R))$ $\quad P \rightarrow R \leftrightarrow \neg P \vee R$

$\leftrightarrow ((\neg P \vee Q) \wedge (\neg P \vee R)) \wedge ((P \vee \neg Q) \wedge (P \vee \neg R))$ distributive law

This is a CNF, as it is a product of elementary sums.

Example 18**:** Obtain a conjunctive normal form of the formula.

$P \rightarrow ((P \rightarrow Q) \wedge \neg (\neg Q \vee \neg P))$

Solution :

$P \rightarrow ((P \rightarrow Q) \wedge \neg (\neg Q \vee \neg P))$ Reasons

$\leftrightarrow \neg P \vee ((P \rightarrow Q) \wedge \neg (\neg Q \vee \neg P))$ $P \rightarrow R \leftrightarrow \neg P V R$

$\leftrightarrow \neg P \vee ((\neg P \vee Q) \wedge \neg (\neg Q \vee \neg P))$ $P \rightarrow R \leftrightarrow \neg P V R$

$\leftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P))$ De Morgan's law

$\leftrightarrow (\neg P \vee (\neg P \vee Q)) \wedge ((\neg P \vee (Q \wedge P)))$ Distributive law

$\leftrightarrow ((\neg P \vee \neg P) \vee Q) \wedge ((\neg P \vee Q) \wedge (\neg P \vee P))$ associative law

$\leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee Q) \wedge T$ Negation law

$\leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee Q)$ identity law

This is a CNF, as it is a product of elementary sums.

Example 19: Find a conjunctive normal of $(Q \vee (P \wedge R)) \wedge \neg ((P \vee R) \wedge Q)$

Solution :

$(Q \vee (P \wedge R)) \wedge \neg ((P \vee R) \wedge Q)$ Reasons

$\leftrightarrow (Q \vee (P \wedge R)) \wedge (\neg (P \vee R) \vee \neg Q)$ De Morgan's law

$\leftrightarrow (Q \vee (P \wedge R)) \wedge ((\neg (P \wedge \neg R) \vee \neg Q)$ De Morgan's law

$\leftrightarrow (Q \vee P) \wedge (Q \vee R) \wedge (\neg P \vee \neg Q) \wedge (\neg R \vee \neg Q)$ Distributive law

Example 20**:** Show that the formula $Q \wedge (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology, by obtaining a conjunctive normal form, of the formula :

Solution**:** We first obtain a CNF of the given formula

$Q \wedge (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

$Q \vee ((P \vee \neg P) \wedge \neg Q$ using the distributive law

$(Q \vee (P \vee \neg P)) \wedge (Q \vee \neg Q)$ again by distributive law

$(Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)$

This is a CNF, as $Q \vee P \vee \neg P$, $Q \vee \neg Q$ are elementary sums.

A CNF is identically true (i.e.,) a tautology.

Min terms

Let P and Q be two statement variables. Construct all possible formula which consist of conjunctions of P or its negation and conjuctions of Q or its negation. None of the formula should contain both a variable and its negation. Delete a formula if it is the commutative of any one of the remaining formulae. Such conjunctions of P and Q are called the min terms of P and Q.

Example 21:

Minterms of P and Q are $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$ and $\neg P \wedge \neg Q$

Note:

(i) $P \wedge Q$ or $Q \wedge P$ is included but not both.

(ii) $P \wedge \neg P$ and $Q \wedge \neg Q$ are not allowed.

(iii) No two minterms are equivalent.

(iv) Each minterm has the truth value T for exactly one combination of the truth values of the variables P and Q.

Minterms of P and Q.

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	$\boxed{T}$	F	F
F	T	F	F	$\boxed{T}$	F
F	F	F	F	F	$\boxed{T}$

Note :

1. No two minterms are equivalent.

2. Minterms for three variables P, Q, R are

$P \wedge Q \wedge R$, $P \wedge Q \wedge \neg R$, $P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$, $\neg p \wedge Q \wedge R$, $\neg P \wedge Q \wedge \neg R$, $\neg P \wedge \neg Q \wedge R$, $\neg P \wedge \neg Q \wedge \neg R$.

For a given formula an equivalent formula consisting of disjunctions of minterms only is known as its principal disjunctive normal form or the sum of products normal form

Principal Disjunctive Normal Form (PDNF)

The sum of products normal form

A formula which is equivalent to a given formula and which consists of sum of its min terms is called "principal disjunctive normal form" (or) "sum of product of canonical form" of the given formula. Construction of PDNF without truth tables:

(i) to replace conditionals and biconditionals by their equivalent formula involving $\wedge$, $\vee$, $\neg$ only.

(ii) to use De Morgan's laws and distributive laws.

(iii) to drop any elementary product which is a contradiction.

(iv) to obtain minterms in the disjunctions by introducing missing factors.

(v) to delete identical minterms keeping only one, that appear in the disjunctions.

Maxterms

For a given number of variables, the maxterm consists of disjunctions in which each variable or its negation, but not both, appears only once.

Remarks:

(i) The max terms are the duals of minterms.

(ii) Either from the duality principle or directly from the truth tables, it can be ascertained that each of the maxterms has the truth value F for exactly one combination of the truth values of the variables.

(iii) Different maxterms have the truth value F for different combinations of the truth values of the variables.

Maxterms of P and Q.

P	Q	$P \vee Q$	$P \vee \neg Q$	$\neg P \vee Q$	$\neg P \vee \neg Q$
T	T	T	T	T	$\boxed{F}$
T	F	T	T	$\boxed{F}$	T
F	T	T	$\boxed{F}$	T	T
F	F	F	T	T	T

Principal Conjunctive Normal Form (or) Product - of - sums canonical form :

An equivalent formula consisting of conjunctions of max terms only is known as its principal conjunctive normal form or the product-of-sums canonical form.

Remark :

Every formula which is not a tautology has an equivalent PCNF which is unique except for the rearrangement of the factors in the maxterms as well as in the conjunctions.

Note: 1. To find pcnf from the pdnf

$S \leftrightarrow (P\wedge Q) \vee (\neg P\wedge Q)$ (pdnf)

(i) Write negation S that is nothing but the disjunction of the remaining minterms.

The minterms of P and Q are

$P\wedge Q,\ P\wedge\neg Q,\ \neg P\wedge Q,\ \neg P\wedge\neg Q$

Given : $P\wedge Q,\ \neg P\wedge Q$

Remaining term: $P\wedge\neg Q,\ \neg P\wedge\neg Q$

$\neg S \leftrightarrow (P\wedge\neg Q) \vee (\neg P\wedge\neg Q)$

(ii) Negation of the disjunctive normal form $\neg S$ apply duality principle.

$\neg(\neg S) \leftrightarrow (\neg P\vee Q) \wedge (P\vee Q)$

$S \leftrightarrow (\neg P\vee Q) \wedge (P\vee Q)$ (pcnf)

Note 2. To find pdnf from the pcnf.

$S \leftrightarrow (P\vee Q) \wedge (\neg P\vee Q)$ (pcnf)

(i) Write the negation S that is nothing but the conjunction of the remaining maxterms.

The maxterms of P and Q are

$P\vee Q,\ \neg P\vee Q,\ P\vee\neg Q,\ \neg P\vee\neg Q$

Given: $P\vee Q,\ \neg P\vee Q$

Remaining terms: $P\vee\neg Q,\ \neg P\vee\neg Q$

$\neg S \leftrightarrow (P\vee\neg Q) \wedge (\neg P\vee\neg Q)$

(ii) Negation of the conjunctive normal form by using duality principle.

$\neg(\neg S) \leftrightarrow (\neg P \wedge Q) \vee (P \wedge Q)$ (pdnf)

$S \leftrightarrow (\neg P \wedge Q) \vee (P \wedge Q)$ (pdnf)

Note 3 : $S \leftrightarrow (P\wedge Q) \vee (\neg P\wedge Q) \vee (\neg P\wedge\neg Q) \vee (P\wedge\neg Q)$

$\neg S \leftrightarrow$ There is no remaining minterms

Pcnf $\leftrightarrow$ no pcnf for the given formula.

Note 4 : $S \leftrightarrow (P \vee Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q)$

$\neg S \leftrightarrow$ There is no remaining maxterms

Pdnf $\leftrightarrow$ no pdnf for the given formula.

Example 22. Obtain the principal disjunctive normal form of

$\neg P \vee Q$ (or) $P \rightarrow Q$. Also find p.c.n.f

Solution**:** Let $S \leftrightarrow \neg P \vee Q$

Method 1. Truth table method

P	Q	$\neg P$	Q	S	Minterms (T)	Maxterms (F)
T	T	F	T	T	$P \wedge Q$	
T	F	F	F	F		$\neg P \vee Q$
F	T	T	T	T	$\neg P \wedge Q$	
F	F	T	F	T	$\neg P \wedge \neg Q$	

$S \leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$ [(p.d.n.f) = disjunctions of minterms]

$S \leftrightarrow \neg P \vee Q$ [p.c.n.f = conjunction of maxterms]

Method 2:

Let $S \leftrightarrow \neg P \vee Q$

$\leftrightarrow (\neg P \wedge T) \vee (Q \wedge T)$ $P \wedge T \leftrightarrow P$, $\neg P \wedge T \leftrightarrow \neg P$, $Q \wedge T \leftrightarrow Q$

$\leftrightarrow [\neg P \wedge (Q \vee \neg Q)] \vee [Q \wedge (P \vee \neg P)]$ $Q \vee \neg Q \leftrightarrow T$, $P \vee \neg P \leftrightarrow T$

$\leftrightarrow [(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)] \vee [(Q \wedge P) \vee (Q \wedge \neg P)]$ distributive law.

$\leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

$\leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P)$ $\neg P \wedge Q \leftrightarrow Q \wedge \neg P$

$S \leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$ p.d.n.f $P \wedge Q \leftrightarrow Q \wedge P$

Here we have used 3 minterms out of 4 to form p.d.n.f

The minterms of P and Q are

$P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$

Here $P \wedge Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$ are in S.

$\neg S$ is nothing but the remaining minterms.

$\neg S \leftrightarrow P \wedge \neg Q$

$\neg(\neg S) \leftrightarrow \neg P V Q$ [Apply duality principle]

i.e., $S \leftrightarrow \neg P V Q$ [p.c.n.f]

Method 2**:**

$S \leftrightarrow P \leftrightarrow Q$

$\leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

$(P \leftrightarrow Q) \leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

$\leftrightarrow (\neg P V Q) \wedge (\neg Q V P)$

$P \rightarrow Q \leftrightarrow \neg P V Q$

$\leftrightarrow (\neg P \wedge \neg Q) \vee (\neg P \wedge P) \vee (Q \wedge \neg Q) \vee (Q \wedge P)$ extended distributive lave

$\leftrightarrow (\neg P \wedge \neg Q) \vee (Q \wedge P)$ $\neg P \wedge P$, $Q \wedge \neg Q$ terms dropped.

$\leftrightarrow (\neg P \wedge \neg Q) \vee (P \wedge Q)$ $P \wedge Q \leftrightarrow Q \wedge P$

$S \leftrightarrow (\neg P \wedge \neg Q) \vee (P \wedge Q)$ (pdnf)

$\neg S \leftrightarrow$ The remaining minterms

$\leftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q)$

Since the minterms of P and Q are

$P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$

$\neg(\neg S) \leftrightarrow$ Apply duality principle to $\neg S$

$S \leftrightarrow (P V \neg Q) \wedge (\neg P V Q)$ (pcnf)

Example 23. Without constructing the truth table obtain the product of sums canonical form of the formula.

$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$. Hence find the sum of products canonical form**.**

Solution: Let $S \leftrightarrow (P \rightarrow R) \wedge (Q \leftrightarrow P)$

Method 1. Truth table method

P Q R	$\neg P$	$\neg P \to R$	$Q \Leftrightarrow P$	S	minterm	maxterm
T T T	F	T	T	T	$P \wedge Q \wedge R$	--
T T F	F	T	T	T	$P \wedge Q \wedge \neg R$	--
T F T	F	T	F	F	--	$\neg P \vee Q \vee \neg R$
F T T	T	T	F	F	--	$P \vee \neg Q \vee \neg R$
T F F	F	T	F	F	--	$\neg P \vee Q \vee R$
F F T	T	T	T	T	$\neg P \wedge \neg Q \wedge R$	--
F T F	T	F	F	F	--	$P \vee \neg Q \vee R$
F F F	T	F	T	F	--	$P \vee Q \vee R$

S ⇔ (P^Q^R) V (P^Q^┐R) V (┐p^┐Q^R) (pdnf)

S ⇔ (┐P v Q v ┐R) Λ (PV┐QV┐R) Λ (┐PVQVR) Λ (PV┐QVR) Λ (PVQVR)   (pcnf)

Method 2.

Let S ↔ (P→R) ^ (Q↔P)

[┐(┐P) VR] ^ [(Q→P) ^ (P→Q)]

P→Q ↔ ┐PVQ, Q↔P ↔ (Q→P) ^ (P→Q)

↔ (PVR) ^ [(┐QVP) ^ (┐PVQ)]

↔ (PVR) ^ (┐QVP) ^ (┐PVQ)

↔ [(PVR) v F] ^ [(┐QVP) v F] ^ [(┐PVQ) VF]   [PVF ↔ P]

↔ [(PVR) v (Q^┐Q) ^ [(┐QVP) V (R Λ ┐R)] ^ [(┐PVQ) V (R Λ ┐R)]

[QΛ┐Q↔F, RΛ┐R↔F]

↔ [(PVRVQ) ^ (PVRV┐Q)] ^ [(┐QVPVR) ^ (┐QVPV┐R) ^ [(┐PVQVR) ^ (┐PVQV┐R)]   by distributive law

↔ [(PVRVQ) ^ (PVRV┐Q) ^ (┐QVPVR) ^ (┐QVPV┐R) ^ (┐PVQVR) ^ (┐PVQV┐R)

PVRV┐Q ↔ ┐QVPVR

S ↔ [(PVQVR) ^ (PV┐QVR) ^ (PV┐QV┐R) ^ (┐PVQVR) ^ (┐PVQV┐R)   pcnf

┐S ↔ The remaining maxterms of P, Q and R

Maxterms of P, Q, R are

(PVQVR), (PvQV┐R), (Pv┐QvR), (┐PVQVR), (Pv┐Qv┐R), (┐PvQv┐R), (┐Pv┐QvR), (┐Pv┐Qv┐R)

$\neg S \leftrightarrow (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

$\neg\neg (S) \leftrightarrow$ Apply duality principle to S

$S \leftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$ pdnf

2.11 Rule of inference

Table of Rule of inference

Rule of inference	Description
Modus Ponens (MP)	If P implies Q, and P is true, then Q is true.
Modus Tollens (MT)	If *P* implies *Q*, and *Q* is false, then *P* is false.
Hypothetical Syllogism (HS)	If P implies Q and Q implies R, then P implies R.
Disjunctive Syllogism (DS)	If P or Q is true, and P is false, then Q is true.
Addition (Add)	If *P* is true, then *P* or *Q* is true.
Simplification (Simp)	If P and Q are true, then P is true
Conjunction (Conj)	If P is true and Q is true, then P and Q are true.

Addition

If P is a premise, we can use Addition rule to derive $P \vee Q$

P

∴P∨Q

Example

Let P be the proposition, “He studies very hard” is true

Therefore − "Either he studies very hard Or he is a very bad student." Here Q is the proposition “he is a very bad student”.

Conjunction

If P and Q are two premises, we can use Conjunction rule to derive P∧Q

P Q

∴P∧Q

Example

Let P − “He studies very hard”

Let Q − “He is the best boy in the class”

Therefore − "He studies very hard and he is the best boy in the class"

Simplification

If P∧Q is a premise, we can use Simplification rule to derive P.

P∧Q

∴P

Example

"He studies very hard and he is the best boy in the class", P∧Q

Therefore − "He studies very hard"

Modus Ponens

If P and P→Q are two premises, we can use Modus Ponens to derive Q.

P→Q

P

∴Q

Example

"If you have a password, then you can log on to facebook", P→Q

"You have a password", P

Therefore − "You can log on to facebook"

Modus Tollens

If P→Q and ¬Q are two premises, we can use Modus Tollens to derive ¬P

.P→Q

¬Q∴

¬P

Example

"If you have a password, then you can log on to facebook", P→Q

"You cannot log on to facebook", ¬Q

Therefore − "You do not have a password "

Disjunctive Syllogism

If ¬P and P∨Q are two premises, we can use Disjunctive Syllogism to derive Q.

¬P

P∨Q

∴Q

Example

"The ice cream is not vanilla flavored", ¬P

"The ice cream is either vanilla flavored or chocolate flavored", P∨Q

Therefore − "The ice cream is chocolate flavored”

Hypothetical Syllogism

If P→Q and Q→R are two premises, we can use Hypothetical Syllogism to derive P→R

P→Q

$Q \rightarrow R$

$\therefore P \rightarrow R$

Example

"If it rains, I shall not go to school", $P \rightarrow Q$

"If I don't go to school, I won't need to do homework", $Q \rightarrow R$

Therefore − "If it rains, I won't need to do homework"

Constructive Dilemma

If $(P \rightarrow Q) \wedge (R \rightarrow S)$and $P \vee R$ are two premises, we can use constructive dilemma to derive $Q \vee S$

.$(P \rightarrow Q) \wedge (R \rightarrow S)$

$P \vee R$

$\therefore Q \vee S$

Example

"If it rains, I will take a leave", $(P \rightarrow Q)$

"If it is hot outside, I will go for a shower", $(R \rightarrow S)$

"Either it will rain or it is hot outside", $P \vee R$

Therefore − "I will take a leave or I will go for a shower"

Destructive Dilemma

If $(P \rightarrow Q) \wedge (R \rightarrow S)$ and $\neg Q \vee \neg S$ are two premises, we can use destructive dilemma to derive $\neg P \vee \neg R$

$(P \rightarrow Q) \wedge (R \rightarrow S)$

$\neg Q \vee \neg S$

$\therefore \neg P \vee \neg R$

Example

"If it rains, I will take a leave", $(P \rightarrow Q)$

"If it is hot outside, I will go for a shower", $(R \rightarrow S)$

"Either I will not take a leave or I will not go for a shower", $(\neg Q \vee \neg S)$

Therefore − "Either it does not rain or it is not hot outside

Example 24. Given Premises: $\begin{cases} \neg A \to (C \wedge D) \\ A \to B \\ \neg B \end{cases}$

Prove: C.

Solution:

1.	$A \to B$	Premise
2.	$\neg B$	Premise
3.	$\neg A$	Modus tollens (1,2)
4.	$\neg A \to (C \wedge D)$	Premise
5.	$C \wedge D$	Modus ponens (3,4)
6.	C	Decomposing a conjunction (5)

Example 25. Given Premises: $\begin{cases} P \wedge Q \\ P \to \neg(Q \wedge R) \\ S \to R \end{cases}$.

Prove: $\neg S$.

Solution:

1.	$P \wedge Q$	Premise
2.	P	Decomposing a conjunction (1)
3.	Q	Decomposing a conjunction (1)
4.	$P \to \neg(Q \wedge R)$	Premise
5.	$\neg(Q \wedge R)$	Modus ponens (3,4)
6.	$\neg Q \vee \neg R$	DeMorgan (5)
7.	$\neg R$	Disjunctive syllogism (3,6)
8.	$S \to R$	Premise
9.	$\neg S$	Modus tollens (7,8) $\square$

Example 26. Given Premises: $\begin{cases} \neg(A \vee B) \to C \\ \neg A \\ \neg C \end{cases}$.

Prove: B.

Solution:

1.	$\neg(A \vee B) \rightarrow C$	Premise
2.	$\neg C$	Premise
3.	$A \vee B$	Modus tollens (1,2)
4.	$\neg A$	Premise
5.	B	Disjunctive syllogism (3,4)

2.12 Quantifiers

Quantifier is used to quantify the variable of predicates. It contains a formula, which is a type of statement whose truth value may depend on values of some variables. When we assign a fixed value to a predicate, then it becomes a proposition. In another way, we can say that if we quantify the predicate, then the predicate will become a proposition. So quantify is a type of word which refers to quantifies like **"all"** or **"some"**.

Universal Quantifiers

Using the universal quantifiers, we can easily express these statements. The universal quantifier symbol is denoted by the **∀**, which means "**for all**".

Suppose P(x) is used to indicate predicate, and D is used to indicate the domain of x. The universal statement will be in the form **"$\forall x \in D, P(x)$".** The main purpose of a universal statement is to form a proposition. In the quantifiers, the domain is very important because it is used to decide the possible values of x.

Example 27**:** Suppose P(x) indicates a predicate where "x must take an electronics course" and Q(x) also indicates a predicate where "x is an electrical student". Now we will find the universal quantifier of both predicates.

Solution**:** Suppose the students are from ABC College. For both predicates, the universe of discourse will be all ABC students.

The statements can be: "Every electrical student must take an electronics course". The following syntax is used to define this statement:

$\forall x(Q(x) \Rightarrow P(x))$

Example 28**:** Suppose P(x) indicates a predicate where "x is a square" and Q(x) also indicates a predicate where "x is a rectangle". Now we will find the universal quantifier of these predicates.

Solution:

The statement must be:

$\forall x$ (x is a square $\Rightarrow$ x is a rectangle), i.e., "all squares are rectangles." The following syntax is used to describe this statement:

$\forall xP(x) \Rightarrow Q(x)$

Existential Quantifiers

Using existential quantifiers, we can easily express these statements. The existential quantifier symbol is denoted by the **∃**, which means **"there exists"**.

Suppose P(x) is used to indicate predicate, and D is used to indicate the domain of x. The existential statement will be in the form "**$\exists x \in D$** such that P(x)".

The main purpose of an existential statement is to form a proposition. The sentence **$\exists xP(x)$** will be **true** if and only if P(x) is true for at least one x in D. The statement **$\exists xP(x)$** will be **false** if and only if P(x) is false for all x in D.

Example 29**:** Suppose P(x) contains a statement "$x > 4$". Now we will find the truth value of this statement.

Solution**:**

This statement is false for all real number which is less than 4 and true for all real numbers which are greater than 4.

This statement is false for $x= 6$ and true for $x = 4$. Now we will compare the above statement with the following statement. So

$\exists xP(x)$ is true

Self-assessment questions

1. Explain logical connectives with truth table
2. Summarize Statement Formulas
3. Write the rules of the Well-Formed Formulas
4. Write in detail about Quine’s method
5. Elucidate CNF and DNF
6. Neatly explain PCNF with example

7. Neatly explain PDNF with example

8. Recall Quantifiers

9. Elaborate Universal Quantifiers

10. Analyze Existential Quantifiers

Let us sum up

Mathematical logic is a branch of mathematics that deals with formal systems, proofs, and the foundations of mathematics. It encompasses several key areas, each addressing different aspects of reasoning and formal structures**.** Truth tables are used to determine the truth values of propositions and logical connectives. They systematically explore all possible combinations of truth values for the involved propositions and provide a clear method for analyzing logical statements**.** Mathematical logic provides the rigorous foundation for much of modern mathematics and computer science, offering tools and frameworks for formal reasoning, proof construction, and understanding the limits of computation and formal systems.

Check your progress

The compound propositions p and q are called logically equivalent if ________ is a tautology.
a) $p \leftrightarrow q$
b) $p \rightarrow q$
c) $\neg (p \lor q)$
d) $\neg p \lor \neg q$

2. $p \rightarrow q$ is logically equivalent to ________
a) $\neg p \lor \neg q$
b) $p \lor \neg q$
c) $\neg p \lor q$
d) $\neg p \land q$

3. $p \lor q$ is logically equivalent to ________
a) $\neg q \rightarrow \neg p$
b) $q \rightarrow p$
c) $\neg p \rightarrow \neg q$
d) $\neg p \rightarrow q$

4. $\neg (p \leftrightarrow q)$ is logically equivalent to ________
a) $q \leftrightarrow p$
b) $p \leftrightarrow \neg q$
c) $\neg p \leftrightarrow \neg q$
d) $\neg q \leftrightarrow \neg p$

5. $p \wedge q$ is logically equivalent to ________
a) $\neg (p \to \neg q)$
b) $(p \to \neg q)$
c) $(\neg p \to \neg q)$
d) $(\neg p \to q)$

6. Which of the following statement is correct?
a) $p \vee q \equiv q \vee p$
b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
c) $(p \vee q) \vee r \equiv p \vee (q \vee r)$
d) All of mentioned

7. $p \leftrightarrow q$ is logically equivalent to ________
a) $(p \to q) \to (q \to p)$
b) $(p \to q) \vee (q \to p)$
c) $(p \to q) \wedge (q \to p)$
d) $(p \wedge q) \to (q \wedge p)$

8. $(p \to q) \wedge (p \to r)$ is logically equivalent to ________
a) $p \to (q \wedge r)$
b) $p \to (q \vee r)$
c) $p \wedge (q \vee r)$
d) $p \vee (q \wedge r)$

9. $(p \to r) \vee (q \to r)$ is logically equivalent to ________
a) $(p \wedge q) \vee r$
b) $(p \vee q) \to r$
c) $(p \wedge q) \to r$
d) $(p \to q) \to r$

10. $\neg (p \leftrightarrow q)$ is logically equivalent to ________
a) $p \leftrightarrow \neg q$
b) $\neg p \leftrightarrow q$
c) $\neg p \leftrightarrow \neg q$
d) $\neg q \leftrightarrow \neg p$

Unit summary

Mathematical logic is the study of formal systems used to understand the nature of mathematical reasoning. At its core, it involves propositional logic, where we work with basic statements (propositions) that can be true or false, using logical connectives like AND, OR, and NOT to build complex expressions. Truth tables help determine the truth values of these expressions. Predicate logic extends this by introducing predicates, which are functions that return true or false based on their inputs, and quantifiers like "for all" and "there exists" to make statements about elements within a domain.

In mathematical logic, proof techniques are fundamental. Direct proofs use logical steps to demonstrate a statement, while indirect proofs, including proof by contradiction and contrapositive, work by assuming the negation of what needs to be proven or its logical converse. Mathematical induction is another technique, particularly for proving statements about natural numbers.

Formal systems involve creating a structured framework with axioms (basic truths) and rules of inference to derive theorems. A system's consistency ensures no contradictions can be derived, while completeness guarantees that all true statements can be proven within the system. Gödel's Incompleteness Theorems reveal limitations, showing that no sufficiently powerful system can be both complete and consistent.

Model theory studies the relationship between formal languages and their interpretations (models). A model satisfies a formula if the formula holds true in that model, and elementary equivalence describes when two models satisfy the same first-order sentences. Additionally, set theory and logic intersect through concepts such as sets, relations, and functions, forming the foundation for more complex logical and mathematical reasoning.

Glossary

Axiom: A fundamental statement or proposition that is assumed to be true without proof, serving as a basis for a logical system.

Contrapositive: The statement formed by negating both the hypothesis and conclusion of a conditional statement and then reversing them..

Decidability: A property of a logical system or problem where there exists an algorithm that can determine the truth or falsity of any statement within the system.

Formal Language: A set of strings of symbols defined by formal rules. In mathematical logic, it includes syntactical structures such as propositional and predicate logic.

Implication: A logical operation represented by $p \to q$ and $p \to q$ that is true if q is true whenever p is true, and false only when p is true and q is false.

Model: An interpretation of a formal language where the statements of the language are evaluated as true or false. A model satisfies a formula if the formula is true under that interpretation.

Negation: A logical operation represented by ¬p\neg p¬p that inverts the truth value of a proposition ppp. If ppp is true, ¬p\neg p¬p is false, and vice versa.

Predicate: A function that returns a true or false value depending on the inputs. In predicate logic, predicates are used to form statements about objects.

Predicate Logic: An extension of propositional logic that includes quantifiers and predicates, allowing more complex statements about objects and their properties.

Proposition: A declarative statement that can be either true or false, but not both.

Proof by Contradiction: A proof technique where one assumes the negation of the statement to be proven and derives a contradiction, thereby proving the original statement.

Proof by Induction: A method of proof that establishes the truth of a statement for all natural numbers by proving a base case and an inductive step.

Quantifier: Symbols used in predicate logic to express the extent to which a predicate applies. Common quantifiers are the universal quantifier ∀\forall∀ (for all) and the existential quantifier ∃\exists∃ (there exists).

Syntactic: Pertaining to the structure and rules of formation of expressions within a formal system, as opposed to semantic meanings.

Tautology: A statement that is true in every possible interpretation or scenario

Truth Table: A table used to determine the truth value of logical expressions by listing all possible truth values for their components and showing the resulting truth value of the entire expression.

Suggested readings

1. Discrete mathematics for computer science

   Book by Gary Haggard

Unit-III

3.1 Recurrence relation

3.2 Solving Recurrence Relations

3.3 Solving recurrence relation by iteration

3.4 Solving Linear Homogeneous Recurrence Relations of Order Two

3.5 Solving Linear Non homogeneous Recurrence Relations

3.6 Permutation

3.7 Combination

Unit-III

3.1 Recurrence relation

Recurrence relation is an equation which represents a sequence based on some rule. It helps in finding the subsequent term (next term) dependent upon the preceding term (previous term). If we know the previous term in a given series, then we can easily determine the next term.

Formulation

Recurrence Relation Formula

Let us assume x_n is the nth term of the series. Then the recurrence relation is shown in the form of;

$x_n + 1 = f(x_n)$; n>0

Where $f(x_n)$ is the function.

We can also define a recurrence relation as an expression that represents each element of a series as a function of the preceding ones.

$x_n = f(n, x_{n-1})$; n>0

To write the recurrence relation of first-order, say order k, the above formula can be represented as;

$x_n = f(n, x_{n-1}, x_{n-2}, \ldots\ldots, x_{n-k})$; n-k>0

Examples of Recurrence Relation

In Mathematics, we can see many examples of recurrence based on series and sequence pattern. Let us see some of the examples here.

1.Factorial Representation

We can define the factorial by using the concept of recurrence relation, such as;

n!=n(n-1)! ; n>0

When n = 0,

0! = 1 is the initial condition.

To find the further values we have to expand the factorial notation, where the succeeding term is dependent on the preceding one.

2.Fibonacci Numbers

In Fibonacci numbers or series, the succeeding terms are dependent on the last two preceding terms. Therefore, this series is the best example of recurrence. As we know from the definition of the Fibonacci sequence,

$$\mathbf{F_n = F_{n-1} + F_{n-2}}$$

Now, if we take the initial values;

$F_0 = 0$ and $F_1 = 1$

So, $F_2 = F_1 + F_0 = 0 + 1 = 1$

In the same way, we can find the next succeeding terms, such as;

$F_3 = F_2 + F_1$

$F_4 = F_3 + F_2$

And so on.

Thus, the Fibonacci series is given by;

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …∞

Note:

In the same way, there are other examples of recurrence such as a logical map, binomial coefficients where the same concept is applicable. Also, arithmetic and geometric series could be called a recurrence sequence.

3.2 Solving Recurrence Relations

To solve given recurrence relations we need to find the initial term first.

Suppose we have been given a sequence; $\mathbf{a_n = 2a_{n-1} - 3a_{n-2}}$

- Now the first step will be to check if initial conditions $a_0 = 1$, $a_1 = 2$, gives a closed pattern for this sequence.
- Then try with other initial conditions and find the closed formula for it.
- The result so obtained after trying different initial condition produces a series.

- Check the difference between each term, it will also form a sequence.
- We need to add all the terms of the new sequence, to understand which sequence is formed
- After understanding the pattern we can now identify the initial condition of the recurrence relation

Example 1:

Solve the recurrence relation $a_n = a_{n-1} – n$ with the initial term $a_0 = 4$.

Solution**:** Let us write the sequence based on the equation given starting with the initial number.

The sequence will be 4,5,7,10,14,19,…..

Now see the difference between each term.

$a_1 – a_0 = 1$

$a_2 – a_1 = 2$

$a_3 – a_2 = 3$

………

$a_n – a_{n-1} = n$

and so on.

Now adding all these equations both at the right-hand side, we get;

$1 +2+3+4+…..n = 1/2 [n(n+1)]$

Whereas on the left-hand side we get;

$(a_1 – a_0) + (a_2 – a_1)+ (a_3 – a_2)+ …….+ (a_{n-1} – a_{n-2})+ (a_n – a_{n-1})$

So you can see, all the terms get cancelled but $– a_0$ and a_n

Therefore, $a_n – a_0 = 1/2 [n(n+1)]$

or

$a_n = 1/2 [n(n+1)] + a_0$

Hence, the solution to the recurrence relation with initial condition $a_0 = 4$, is;

$a_n = 1/2 [n(n+1)] + 4$

3.3 Solving recurrence relation by iteration

The iteration method is a "brute force" method of solving a recurrence relation. The general idea is to iteratively substitute the value of the recurrent part of the equation until a pattern (usually a summation) is noticed, at which point the summation can be used to evaluate the recurrence.

Example 2:

Suppose we have the following recurrence relation:

$$T(1) = \Theta(1)$$

$$T(n) = c_1 + 2(T(n-1))$$

then:

$$T(n) = c_1 + 2(\qquad\qquad)$$

$$T(n) = c_1 + 2(\qquad\qquad\qquad\qquad)$$

$$T(n) = c_1 + 2(c_1 + 2(\qquad\qquad\qquad\qquad))$$

$$T(n) = c_1 + 2(c_1 + 2(c_1 + 2(\qquad\qquad\qquad\qquad)))$$

Each iteration, the recurrence is replaced with its value as established by the original recurrence relation. Now that we've done a few iterations,

let's simplify and see if there is a recognizable pattern.

$$T(n) = c_1 + 2(c_1 + 2(c_1 + 2(c_1 + 2(T(n-4)))))$$

$$T(n) = c_1 + 2c_1 + 4(c_1 + 2(c_1 + 2(T(n-4))))$$

$$T(n) = c_1 + 2c_1 + 4c_1 + 8(c_1 + 2(T(n-4)))$$

$$T(n) = c_1 + 2c_1 + 4c_1 + 8c_1 + 16(T(n-4))$$

$$T(n) = 2^0 c_1 + 2^1 c_1 + 2^2 c_1 + 2^3 c_1 + 2^4 (T(n-4))$$

There definitely seems to be a pattern here. Each iteration we're adding a 2^i term; where i is the number of iterations that we have made.

Now the question is: When is this going to stop?

From the original problem we know that: $T(1) = \Theta(1)$. We can say that:
$T(n-i) = 1$ when $i = n-1$

Now we can write our simplified equation in terms of i .

$$T(n) = 2^i T(n-i) + 2^{i-1}c_1 + 2^{i-2}c_1 + \ldots + 2^0 c_1$$

Now $n - i = n - (n-1) = 1$ and we know that $T(1) = \Theta(1)$.

Also any constant is $\Theta(1)$. So we can re-write our equation as:

$$T(n) = 2^i \Theta(1) + 2^{i-1}\Theta(1) + 2^{i-2}\Theta(1) + \ldots + 2^0 \Theta(1)$$
$$T(n) = \Theta(1)\left(2^i + 2^{i-1} + 2^{i-2} + \ldots + 2^0\right)$$
$$T(n) = \Theta(1)\sum_{i=0}^{n-1} 2^i$$
$$T(n) = \Theta(1)\frac{2^{((n-1)+1)} - 1}{2-1}$$
$$T(n) = \Theta(1)\left(2^n - 1\right)$$
$$T(n) = \Theta\left(2^n\right)$$

So, the time complexity of this recurrence relations is $\Theta\left(2^n\right)$.

Example 3:

Suppose we have the following recurrence relation:

$$T(1) = \Theta(1)$$
$$T(n) = n^3 + 2(T(n/2))$$

then:

$$T(n) = n^3 + 2(T(n/2))$$
$$T(n) = n^3 + 2\left((n/2)^3 + 2(T(n/4))\right)$$
$$T(n) = n^3 + 2\left((n/2)^3 + 2\left((n/4)^3 + 2(T(n/8))\right)\right)$$
$$T(n) = n^3 + 2\left((n/2)^3 + 2\left((n/4)^3 + 2\left((n/8)^3 + 2(T(n/16))\right)\right)\right)$$

Each iteration, the recurrence is replaced with its value as established by the original recurrence relation. Notice that each iteration the n^3 term is replaced with $(n/2)^3$, then $(n/4)^3$, and so on.

Now that we've done a few iterations, let's simplify and see if there is a recognizable pattern.

$$T(n) = n^3 + 2\Big((n/2)^3 + 2\Big((n/4)^3 + 2\Big((n/8)^3 + 2\big(T(n/16)\big)\Big)\Big)\Big)$$

$$T(n) = n^3 + 2(n/2)^3 + 4\Big((n/4)^3 + 2\Big((n/8)^3 + 2\big(T(n/16)\big)\Big)\Big)$$

$$T(n) = n^3 + 2(n/2)^3 + 4\Big((n/4)^3 + 2\Big((n/8)^3 + 2\big(T(n/16)\big)\Big)\Big)$$

$$T(n) = n^3 + 2(n/2)^3 + 4(n/4)^3 + 8\Big((n/8)^3 + 2\big(T(n/16)\big)\Big)$$

$$T(n) = n^3 + 2(n/2)^3 + 4(n/4)^3 + 8(n/8)^3 + 16\big(T(n/16)\big)$$

$$T(n) = 2^0\left(\frac{n}{2^0}\right)^3 + 2^1\left(\frac{n}{2^1}\right)^3 + 2^2\left(\frac{n}{2^2}\right)^3 + 2^3\left(\frac{n}{2^3}\right)^3 + 2^4\left(T\left(n/2^4\right)\right)$$

$$T(n) = \frac{n^3}{4^0} + \frac{n^3}{4^1} + \frac{n^3}{4^2} + \frac{n^3}{4^3} + 2^4\left(T\left(n/2^4\right)\right)$$

There's a definite pattern in all but the last term of the equation, and the last term seems to be related by the power that 2 is raised to.

Now the question is: When is this going to stop?

From the original problem we know that: $T(1) = \Theta(1)$. We can say that:

$$T\left(n/2^i\right) = 1 \text{ when } i = \lg n$$

Now we can write our simplified equation in terms of i .

$$T(n) = 2^i\left(T\left(n/2^i\right)\right) + n^3\left(\frac{1}{4^{i-1}} + \frac{1}{4^{i-2}} + \ldots + \frac{1}{4^1} + \frac{1}{4^0}\right)$$

Now since $i = \lg n$, $2^i\left(T\left(n/2^i\right)\right) = 2^{\lg n}\left(T\left(n/2^{\lg n}\right)\right) = n\left(T(1)\right) = \Theta(n)$. We can also recognize the second half of the equation as a geometric series; allowing us to re-write the equation as:

$$T(n) = 2^i\left(T\left(n/2^i\right)\right) + n^3\left(\frac{1}{4^{i-1}} + \frac{1}{4^{i-2}} + \ldots + \frac{1}{4^1} + \frac{1}{4^0}\right)$$

$$T(n) = \Theta(n) + n^3\sum_{i=0}^{\lg n-1}\left(\frac{1}{4}\right)^i$$

Although we could work out the summation, it's much easier to realize that this is a finite geometric series, meaning that the summation will be a constant. Since any constant is $\Theta(1)$, we are left with:

$$T(n) = \Theta(n) + n^3\Theta(1)$$

$$T(n) = \Theta(n) + \Theta(n^3)$$

$$T(n) = \Theta(n^3)$$

So, the time complexity of this recurrence relations is $\Theta(n^3)$.

3.4 Solving Linear Homogeneous Recurrence Relations of Order Two

Solving linear recurrence equations is similar to solving linear differential equations.
an=−4an−1−4an−2.

an=bn⟹b2+4b+4=0.

Thus the solution is an=A(−2)n+Bn(−2)n.

The solution is an=(−1)nn2n**−**1**.**

Example 4:

Consider the following second-order recurrence equation:

T(n) = a$_{1T(n-1) + a2T(n-2)}$

For solving this equation formulate it into a characteristic equation.
Let us rearrange the equation as follows:

T(n) - a$_{1T(n-1) - a2T(n-2) = 0}$

Let, *T(n) = x*^{n}
Now we can say that *T(n-1) = x*$^{n-1}$ and *T(n-2)=x*$^{n-2}$
Now the equation will be:

$x^{n + a1xn-1 + a2xn-2 = 0}$

After dividing the whole equation by x^{n-2} [since x is not equal to 0] we get:

$x^{2 + a1x + a2 = 0}$

We have got our characteristic equation as $x^2+a_1x+a_2=0$
Now, find the roots of this equation.
Three cases may exist while finding the roots of the characteristic equation and those are:

Case 1: Roots of the Characteristic Equation are Real and Distinct
If there are r number of distinct roots for the characteristic equation then it can be said that r number of fundamental solutions are possible. One can take any linear combination of roots to get the general solution of a linear recurrence equation.

If $r_1, r_2, r_3……, r_k$ are the roots of the characteristic equation then the general solution of the recurrence equation will be:

$t_{n = c1r1}^{n + c2r2n + c3r3n +..........+ ckrkn}$

Case 2: Roots of the Characteristic Equation are Real but not Distinct
Let us consider the roots of the characteristic equations are not distinct and the roots r is in the multiplicity of m. In this case, the solution of the characteristic equation will be:

$r_{1 = r}^{n}$

$r_{2 = nr}^{n}$

$r_{3 = n}^{2rn}$

………

$r_{m = n}^{m-1rn}$

Therefore, include all the solutions to get a general solution of the given recurrence equation.

Case 3: Roots of the Characteristic Equation are Distinct but Not Real
If the roots of the characteristic equation are complex, then find the conjugate pair of roots. If r_1 and r_2 are the two roots of a characteristic equation, and they are in conjugate pair with each other it can be expressed as:

$r_{1 = re}^{ix}$

$r_{2 = re}^{-ix}$

The general solution will be:

$t_{n} = r^n(c_{1cos\ nx\ +\ c2sin\ nx)}$

Example 5:

Let’s solve the given recurrence relation:

*T(n) = 7*T(n-1) - 12*T(n-2)*

Let T(n) = x^n
Now we can say that T(n-1) = x^{n-1} and T(n-2)=x^{n-2}

And dividing the whole equation by x^{n-2},

we get:

$x^{2\ -\ 7*x\ +\ 12\ =\ 0}$

3.5 Solving Linear Non homogeneous Recurrence Relations.

A first-order non-homogeneous linear recurrence has the form un=aun−1+p(n) with a∈R$un=aun$−1+$p(n)$ with $a∈R$

The following table shows some examples of the trial solution for p_n:

Non-homogeneous term $d(n)$	Trial solution for p_n
Constant	
$d(n) = a$	$p_n = A$
Polynomials	
$d(n) = n$	$p_n = An + B$
$d(n) = n + a$	$p_n = An + B$
$d(n) = n^2 + n + a$	$p_n = An^2 + Bn + C$
Trigonometric functions	
$d(n) = \sin(an)$	$p_n = A\cos(an) + B\sin(an)$
$d(n) = \cos(an)$	$p_n = A\cos(an) + B\sin(an)$
$d(n) = \sin(an) + \cos(an)$	$p_n = A\cos(an) + B\sin(an)$
Power / exponential functions	
$d(n) = 2^n$	$p_n = A2^n$
$d(n) = -5^n$	$p_n = A5^n$
$d(n) = e^n$	$p_n = Ae^n$
Combination functions	
$d(n) = n + 2^n$	$p_n = (An + B) + C2^n$
$d(n) = (n + a)2^n$	$p_n = (An + B)\,2^n$
$d(n) = n + \sin(an)$	$p_n = (An + B) + C\cos(an) + D\sin(an)$

where a is a constant and A, B, C, D are coefficients to be determined.

Example 6

Find the explicit formula for **$a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = n4^n$** with initial conditions $a_0 = -2$, $a_1 = 0$ and $a_2 = 5$.

Characteristic equation is $r^3 - 7r^2 + 16r - 12 = 0$

$(r - 2)(r - 2)(r - 3) = 0$

Characteristic roots are $r = 2$ (repeated root) and 3

Let $a_n = g_n + p_n$ where $g_n = (\alpha_1 + \alpha_2 n)(2)^n + \alpha_3(3)^n$ and $p_n = (An + B)(4)^n$.

Subs. $p_n = An + B$ into the recurrence relation,

$(An + B)(4)^n = 7[(A(n-1) + B)(4)^{n-1}] - 16[(A(n-2) + B)(4)^{n-2}] + 12[(A(n-3) + B)(4)^{n-3}] + n(4)^n$

$64(An + B)(4)^n = 112(An - A + B)(4)^n - 64(An - 2A + B)(4)^n + 12(An - 3A + B)(4)^n + 64n(4)^n$

Comparing the coefficient of n, Comparing the constant,

$64A = 112A - 64A + 12A + 64$, $64B = -112A + 112B + 128A - 64B - 36A + 12B$

$A = 16$ $B = -80$

Hence, the particular solution $p_n = (16n - 80)(4)^n$

$a_n = (\alpha_1 + \alpha_2 n)(2)^n + \alpha_3(3)^n + (16n - 80)(4)^n$

$a_0 = (\alpha_1 + \alpha_2(0))(2)^0 + \alpha_3(3)^0 + (16(0) - 80)(4)^0$, $-2 = \alpha_1 + \alpha_3 - 80$ — (1)

$a_1 = (\alpha_1 + \alpha_2(1))(2)^1 + \alpha_3(3)^1 + (16(1) - 80)(4)^1$, $0 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 - 256$ — (2)

$a_2 = (\alpha_1 + \alpha_2(2))(2)^2 + \alpha_3(3)^2 + (16(2) - 80)(4)^2$, $5 = 4\alpha_1 + 8\alpha_2 + 9\alpha_3 - 768$ — (3)

$\alpha_1 = 17$, $\alpha_2 = 39/2$ and $\alpha_3 = 61$

So, the explicit formula is $a_n = (17 + 39/2\ n)(2)^n + 61(3)^n + (16n - 80)(4)^n$.

Example 7

Find the explicit formula for $a_n = a_{n-1} + n2^n + 1$ for $n \geq 1$ with initial conditions $a_0 = 1$.

Characteristic equation is $r - 1 = 0$

Characteristic root is $r = 1$

Let $a_n = g_n + p_n$ where $g_n = \alpha(1)^n = \alpha$ and $p_n = (An + B)2^n + Cn + D$

Subs. pn= (An + B)2n + Cn + D into the recurrence relation,

(An+ B)2n + Cn + D = (A(n-1)+ B)2n-1 + C(n-1) + D + n2n + 1

2(An+ B)2n-1 + Cn + D = (An – A + B)2n-1+ 2n2n-1 + Cn – C + D + 1

2A = A+ 2n, 2B = –A + B, C = 0, D = D +1

A = 2n, B = –2n, D = 0

Hence, the particular solution pn = (2n2 – 2n)2n

an = α + (2n2 – 2n)2n

a0 = α + (2(0)2 – 2(0))20

1 = α

So, the explicit formula is an = 1+ (2n2 – 2n)2n.

Example 8

Solve $a_{n+2}+a_{n+1}-6a_n=2^n$ for $n \geq 0$.

Solution

First we observe that the homogeneous problem

$$u_{n+2} + u_{n+1} -6u_n=0$$

has the general solution $u_n=A\ 2^n + B(-3)^n$ for $n \geq 0$ because the associated characteristic equation $\lambda^2+ \lambda -6 =0$ has 2 distinct roots $\lambda_1=2$ and $\lambda_2=-3$.

Since the r.h.s. of the nonhomogeneous recurrence relation is 2^n, if we formally follow the strategy in the previous lecture we would try $v_n=C2^n$ for a particular solution.

But there is a difficulty: $C2^n$ fits into the format of u_n which is a solution of the homogeneous problem.

In other words it can't be a particular solution of the *nonhomogeneous* problem.

This is really because "2" happens to be one of the 2 roots λ_1 and λ_2.

However, we suspect that a particular solution would still have to have 2^n as a factor, so we try $v_n=Cn2^n$.

Substituting it to $v_{n+2}+v_{n+1}-6v_n=2^n$, we obtain

$$C(n+2)2^{n+2}+C(n+1)2^{n+1}-6Cn2^n = 2^n ,$$

i.e. $10C2^n=2^n$ or $C=1/10$.

Hence a particular solution is $v_n=(n/10)2^n$ and the general solution of our nonhomogeneous recurrence relation is

$$a_n=A2^n+B(-3)^n + \frac{n}{10}2^n , \quad n \geq 0 .$$

Example 9

Find the general solution of $f(n+2)-6f(n+1)+9f(n)=5 \times 3^n$, $n \geq 0$.

Solution

Let $f(n)=u_n+v_n$, with u_n being the general solution of the homogeneous problem and v_n a particular solution.

(a) Find u_n: The associated characteristic equation $\lambda^2 -6\lambda + 9 =0$ has a repeated root $\lambda =3$ with multiplicity 2. Hence the general solution of the homogeneous problem

$$u_{n+2}-6u_{n+1}+9u_n=0 , \quad n \geq 0$$

is

$$u_n = (A+Bn)3^n.$$

(b) Find v_n: Since the r.h.s. of the recurrence relation, the nonhomogeneous part, is 5×3^n and 3 is a root of multiplicity 2 of the characteristic equation (i.e. $\mu =3$, $k=0$, $M=2$), we try due to (***) $v_n =B_0 \mu^n \times n^M \equiv Cn^2 3^n$:

we just need to observe that $C3^n$ is of the form 5×3^n and

that the extra factor n^2 is due to $\mu=3$ being a double root of the characteristic equation. Thus

$$\begin{aligned} 5 \times 3^n &= v_{n+2}-6v_{n+1}+9v_n \\ &= C(n+2)^2 3^{n+2}-6C(n+1)^2 3^{n+1}+9Cn^2 3^n \\ &= 18C3^n . \end{aligned}$$

Hence $C=5/18$ and $v_n=(5/18)n^2 3^n$. Therefore our general solution reads

$$f(n) = \left(A + Bn + \frac{5}{18}n^2\right)3^n, \quad n \geq 0 .$$

Example 10

.Find the particular solution of

$$a_{n+4}-5a_{n+3}+9a_{n+2}-7a_{n+1}+2a_n=3 , \ n \geq 0$$

satisfying the initial conditions $a_0 = 2$, $a_1 = -1/2$, $a_2 = -5$, $a_3 = -31/2$.

Solution

We first find the general solution u_n for the homogeneous problem. We then find a particular solution v_n for the nonhomogeneous problem without considering the initial conditions. Then $a_n=u_n+v_n$ would be the general solution of the nonhomogeneous problem.

We finally make use of the initial conditions to determine the arbitrary constants in the general solution so as to arrive at our required particular solution.

(a) Find u_n: Since the associated characteristic equation

$$\lambda^4-5\lambda^3+9\lambda^2-7\lambda+2=0$$

has the sum of all the coefficients being zero, i.e. $1-5+9-7+2=0$, it must have a root $\lambda=1$. After factorising out $(\lambda-1)$ via $\lambda^4-5\lambda^3+9\lambda^2-7\lambda+2=(\lambda-1)(\lambda^3-4\lambda^2+5\lambda-2)$, the rest of the roots will come from $\lambda^3-4\lambda^2+5\lambda-2=0$.

Notice that $\lambda^3-4\lambda^2+5\lambda-2=0$ can again be factorised by a factor $(\lambda-1)$ because $1-4+5-2=0$.

This way we can derive in the end that the roots are

$\lambda_1=1$ with multiplicity $m_1=3$, and

$\lambda_2=2$ with multiplicity $m_2=1$.

Thus the general solutions for the homogeneous problem is

$$u_n = (A+Bn+Cn^2)1^n+D2^n ,$$

or simply $u_n=A+Bn+Cn^2+D2^n$ because $1^n \equiv 1$.

(b) Find v_n: Notice that the nonhomogeneous part is a constant 3 which can be written as 3×1^n when cast into the form of (**), and that 1 is in fact a root of multiplicity 3. In other words, we have in (***) $\mu=1$, $k=0$ and $M=3$. Hence we try a particular solution $v_n=En^3 \cdot 1^n=En^3$. The substitution of v_n into the nonhomogeneous recurrence equations then gives, using a formula in the subsection *Binomial Expansions* in the *Preliminary Mathematics* at the beginning of notes,

$$3 = v_{n+4}-5v_{n+3}+9v_{n+2}-7v_{n+1}+2v_n$$

$$= E(n+4)^3-5E(n+3)^3+9E(n+2)^3-7E(n+1)^3+2En^3$$

$$= E(n^3 + 3n^2 \times 4+3n \times 4^2+4^3) -5E(n^3 + 3n^2 \times 3+3n \times 3^2+3^3)$$

$$+ 9E(n^3 + 3n^2 \times 2+3n \times 2^2+2^3) -7E(n^3 + 3n^2 \times 1+3n \times 1^2+1^3) + 2En^3$$

$$= -6E ,$$

i.e. $E= -1/2$. Hence $v_n= -n^3/2$.

3.6 Permutation

A **permutation** is an arrangement of objects in a definite order. The members or elements of sets are arranged here in a sequence or linear order.

For example, the permutation of set A={1,6} is 2, such as {1,6}, {6,1}. As you can see, there are no other ways to arrange the elements of set A.

Definition of Permutation

Basically Permutation is an arrangement of objects in a particular way or order. While dealing with permutation one should concern about the selection as well as arrangement.

In Short, ordering is very much essential in permutations. In other words, the permutation is considered as an ordered combination.

Representation of Permutation

We can represent permutation in many ways, such as:

- $P(n,k)$
- Pkn
- nPk
- nPk
- Pn,k

Formula

The formula for permutation of n objects for r selection of objects is given by:

$P(n,r) = n!/(n-r)!$

For example, the number of ways 3rd and 4th position can be awarded to 10 members is given by:
$P(10, 2) = 10!/(10-2)! = 10!/8! = (10.9.8!)/8! = 10 \times 9 = 90$

Types of Permutation

Permutation can be classified in three different categories:

- Permutation of n different objects (when repetition is not allowed)
- Repetition, where repetition is allowed
- Permutation when the objects are not distinct (Permutation of multi sets)

Permutation of n different objects

If n is a positive integer and r is a whole number, such that r < n, then P(n, r) represents the number of all possible arrangements or permutations of n distinct objects taken r at a time. In the case of permutation without repetition, the number of available choices will be reduced each time. It can also be represented as: ^nP_r.

$P(n, r) = n(n-1)(n-2)(n-3)$……..upto r factors

$P(n, r) = n(n-1)(n-2)(n-3)$……..$(n – r +1)$

$\Rightarrow P(n,r)=n!(n-r)!$

Here, "^nP_r" represents the "n" objects to be selected from "r" objects without repetition, in which the order matters.

Example 11

How many 3 letter words with or without meaning can be formed out of the letters of the word SWING when repetition of letters is not allowed?

Solution**:** Here n = 5, as the word SWING has 5 letters. Since we have to frame 3 letter words with or without meaning and without repetition, therefore total permutations possible are:

$\Rightarrow (n,r)=5!(5-3)!=5\times4\times3\times2\times12\times1=60$

Permutation when repetition is allowed

We can easily calculate the permutation with repetition. The permutation with repetition of objects can be written using the exponent form.

When the number of object is **"n,"** and we have **"r"** to be the selection of object, then;

Choosing an object can be in n different ways (each time).

Thus, the permutation of objects when repetition is allowed will be equal to,

$n \times n \times n \times \ldots\ldots(r \text{ times}) = n^r$

This is the permutation formula to compute the number of permutations feasible for the choice of "r" items from the "n" objects when repetition is allowed.

Example12

How many 3 letter words with or without meaning can be formed out of the letters of the word *SMOKE* when repetition of words is allowed?

*Solution**:***

The number of objects, in this case, is 5, as the word SMOKE has 5 alphabets.

and r = 3, as 3-letter word has to be chosen.

Thus, the permutation will be:

Permutation (when repetition is allowed) = $5^3 = 125$

Permutation of sets containing indistinguishable elements

Permutation of n different objects when P_1 objects among '**n**' objects are similar, P_2 objects of the second kind are similar, P_3 objects of the third kind are similar ……… and so on, P_k objects of the kth kind are similar and the remaining of all are of a different kind,

Thus it forms a multiset, where the permutation is given as:

$n!$ Divided by $p1!p2!p3\ldots..pn!$

Example 13

1) In how many distinct ways can the letters in the word ITEMS be arranged?
2) In how many distinct ways can the letters in the word STEMS be arranged?
3) In how many distinct ways can the letters in the word SEEMS be arranged?

Solution

1) The word ITEMS has 5 letters. They all are different (unique; distinguishable).

Therefore, there are 5! = 5*4*3*2*1 = 120 distinct ways the letters in the word ITEMS can be arranged.

2) The word STEMS has 5 letters.

There are 4 and only 4 different (distinguishable) letters. Two letters (S) are identical.

Although there are formally 5! = 120 permutations/arrangements, not all of them are distinct/distinguishable.

Namely, in each permutation two identical letters S can be reversed in their positions, but the resulting permutations still represent the same arrangement.

Therefore, every second permutation with reversed S's represents THE SAME ARRANGEMENT.

This is why the entire number of permutations must be divided by 2 to account for this fact. As a result, the final formula for the number of arrangements in this case is $\frac{5!}{2}$ = 60.

3) The word SEEMS has 5 letters. There are 3 and only 3 different (distinguishable) letters. There are two identical letters S and two identical letters E.

Following to the logic of the n 2), we must divide 120 (the total number of formal permutations of 5 symbols) by (2*2) = 4.

As a result, the final formula for the number of arrangements in this case is $\frac{5!}{2 \cdot 2}$ = 30.

Example 14

If you have 6 colored flags: one flag green, 2 flags red and 3 flags blue, How many signals can you form
displaying all 6 flags in different orders?

Solution

The number of different signals is $\frac{6!}{2! \cdot 3!}$ = $\frac{720}{12}$ = 60.

It is equal to the number of all permutations of the 6 flags 6! = 720 divided by 2! to account for two undistinguishable red flags and divided by 3! to account for three indistinguishable blue flags.

Example 15

Suppose we are given 4 identical red flags, 2 identical blue flags, and 3 identical green flags.Find the number m of different signals that can be formed by hanging the 9 flags in a vertical line.

Solution

The number of distinguishable arrangements is $\frac{9!}{4!\cdot 2!\cdot 3!}$ = $\frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{2\cdot 6}$ = 1260 in this case.

In the denominator, the factors 4!, 2! and 3! serve to account for multiplicities of the identical items / flags.

Example 16

In how many different ways can 3 red, 4 yellow and 2 blue bulbs be arranged in a string of Christmas tree lights with 9 sockets?

Solution

The total number of sockets is 9.

There are C_9^3 = $\frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}$ = 84 different ways to select 3 sockets from 9 sockets for 3 red bulbs.There are C_6^4 = $\frac{6\cdot 5}{1\cdot 2}$ = 15 different ways to select 4 sockets from 9-3 = 6 remaining sockets for 4 yellow bulbs.

After that, the sockets for remaining 2 blue bulbs are the remaining 2 sockets, so there is only 1 way to place them (there is no other choice).

In all, there are 84*15 = 1260 ways to arrange the bulbs, according to the Fundamental counting principle.

There is another method to solve the problem.

Among 9 bulbs, we have 3 indistinguishable red bulbs; 4 indistinguishable yellow bulbs and 2 indistinguishable blue bulbs.

So, we apply the formula for distinguishable/indistinguishable permutations and find the number of distinguishable arrangements

$$\frac{9!}{3! \cdot 4! \cdot 2!} = 1260,$$

which gives the same answer.

Example 17

Dada has the password dada112233. She wants to change her password using the same letters in the first four positions and the same numbers in the last 6 positions. In how many ways she can do that?
Solution
So, we have all distinguishable arrangements of the word "dada" in the first 4 positions and all distinguishable arrangements of 6 digits "112233" in the last 6 positions.

The word "dada" has 4 letters in all; of them, there are only 2 distinguishable letters each of the multiplicity 2.

The number of all distinguishable arrangements for letters is $\frac{4!}{2! \cdot 2!} = \frac{24}{4} = 6$ in this case. (2!*2!) in the denominator stays to account for repeated "a" and repeated "d" with their multiplicities.

The word "112233" has 6 digits in all; of them, there are only 3 distinguishable digits each of the multiplicity 2.

The number of all distinguishable arrangements for digits is $\frac{6!}{2! \cdot 2! \cdot 2!} = \frac{720}{8} = 90$ in this case.

(2!*2!*2!) in the denominator stays to account for repeated "1"; repeated "2", and repeated "3" with their multiplicities.

Now, to complete the solution, we need simply multiply 6 by 90.

<u>Answer</u>. There are 540 different ways to create the password in a way described in the problem.

Example 18

How many distinguishable permutations are there of the letters in the word WOODCOCK?
Solution

The word WOODCOCK has 8 letters.

Of them, letter "O" has the multiplicity of 3;

letter "C" has the multiplicity of 2;

the rest of letters are unique.

Therefore, the number of all distinguishable permutations (they are also called "distinguishable arrangements") is

$$\frac{8!}{3!\cdot 2!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{(1\cdot 2\cdot 3)\cdot(1\cdot 2)} = 3360.$$ ANSWER

8! counts the number of all possible permutations of 8 letters.

3! in the denominator stays to account for repeating letter "O".

2! in the denominator stays to account for repeating letter "C".

Example 19

How many distinguishable permutations of letters are possible in the word ENGINEERING?
Solution

In all, there are 11 letters in the word.

Of them, letter "E" has multiplicity 3;

letter "N" has multiplicity 3;

letter "G" has multiplicity 2;

letter "I" has multiplicity 2;

the last letter "R" has multiplicity 1.

The number of distinguishable permutations of letters is $\frac{11!}{3! \cdot 3! \cdot 2! \cdot 2!}$ = $\frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 6 \cdot 2 \cdot 2}$ = 277200.

Example 20

In how many ways can 4 pennies, 5 nickels and 3 dimes be distributed among 12 children, if each is to receive one coin ?
Solution

It is the same as to ask How many 12-letter words could be written using 4 letters P, 5 letters N and 3 letters D? (using letters P, N and D to code the pennies, nickels and dimes, respectively).

The answer is $\frac{12!}{4! \cdot 5! \cdot 3!}$ = 27720, using our knowledge about arranging indistinguishable elements.

Example 21

Jenny's house is 4 blocks west and 3 blocks south of a store.
How many ways can Jenny walk from the store to home if she wishes to walk only west and south?
Solution
Represent walking one block west with W and one block south with S.

From the store, there are 4 blocks west and 3 blocks south to the house.

The number of different ways she can go is the number of different ways of ordering the letters W and S in the 4 + 3 = 7-letter word consisting of these two letters W and S.

As we learned it in this lesson, the number of such words is

$\frac{7!}{3! \cdot 4!} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}$ = 35.

So, there are 35 ways Jenny can walk from the store to her home under given restrictions.

Example 22

If a permutation is chosen random from the letters “aaabbbccc”, what is the probability that it begins with at least 2 a’s ?

Solution

(1) We will consider distinguishable arrangements of 9 given letters.

The total number of all such arrangements is $\frac{9!}{3!\cdot 3!\cdot 3!} = \frac{392880}{6\cdot 6\cdot 6} = 1680.$

(2) The number of all distinguishable arrangements starting with 3 a's is

$\frac{6!}{3!\cdot 3!} = \frac{720}{6\cdot 6} = 20.$

(3) The number of all distinguishable arrangements starting with <u>exactly</u> 2 a's is equal (OBVIOUSLY) to the number of all distinguishable arrangements of the 7 (seven letter) "abbbccc", where "a" is not in the first (leftmost) position.

The number of such arrangements is equal to the number of all distinguishable arrangements of these 7 letters (which is $\frac{7!}{3!\cdot 3!} = 140$) $\boxed{\text{MINUS}}$ the number of all those distinguishable arrangements of these 7 letters, where "a" is the first position. The latter number is $\frac{6!}{3!\cdot 3!} = \frac{720}{6\cdot 6} = 20.$

THUS, the number of all distinguishable arrangements starting with <u>exactly</u> 2 a's is equal to 140 - 20 = 120.

(4) Finally, the number of all distinguishable arrangements starting with at least 2 a's is 20 + 120 = 140.

(5) THEREFORE, the probability under the problem's question is $P = \frac{140}{1680} = \frac{14}{168} = \frac{1}{12}$ = 0.08333... = 8.333...%

Example 23

How many distinct permutations of 4 letters from word EAGLES are there?

Solution

The word EAGLES has 6 letters; of them, one letter E is repeated and have a multiplicity 2.

When we analyze the number of different words of 4 letters formed from the given word (symbols),

we should distinct two different cases.

<u>Case 1</u>. All 4 letters in the final word are different. In this case, we have only 5 distinct letters to choose from, (E, A, G, L S);

therefore, the number of possible words to form is 5*4*3*2 = 120 in this case (the order of letters does matter !);

Case 2. In the final word, we have 2 identical letters E and any 2 of the remaining 4 letters. In this case, we can select these two remaining letters by C_4^2 = 6 different ways and we can arrange then 4 letters with two repeating undistinguishable Es by

$\frac{4!}{2!} = \frac{24}{2}$ = 12 different distinguishable ways.

Combining everything altogether, we have 6*12 = 72 different words in Case 2.

Cases (1) and (2) are the disjoint sets of words; therefore, the answer to the problem's question is 120 + 72 = 192.

ANSWER. 192 different / (distinguished) words of the length 4 can be formed.

Example 24

How many different 4-letter arrangements can be formed from the letters in the word WESTINGHOUSE ?
Solution
The given word consists of 12 letters. Of them, 8 letters are non-repeating; two letters (E and S) are repeating and have multiplicity 2.

So, I first consider 10 unique letters W, E, S, T, I, N, G, H, O, U and will calculate the number of 4-letter arrangements of these letters. All these arrangements consist of non-repeating letters.

Then I will consider all 4-letter distinguishable words having two repeating "E" and no repeating "S";then I will consider all 4-letter distinguishable words having two repeating "S" and no repeating "E";then I will consider all 4-letter distinguishable words having two repeating "E" and two repeating "S".At the end, I will add all these opportunities.

(1) the number of 4-letter arrangements of 10 distinct letters W, E, S, T, I, N, G, H, O, U is

10*9*8*7 = 5040.

(2) the number of all 4-letter distinguishable words having two repeating "E" is

$C_4^2 \cdot 9 \cdot 8 = \left(\frac{4 \cdot 3}{2}\right) \cdot 9 \cdot 8$ = 6*9*8 = 432.

Here $C_4^2 = \frac{4 \cdot 3}{2}$ = 6 is the number ow ways to select two positions for E in the 4-letter word, without looking the order. Next, the factors 9 and 8 are to calculate the number of placing the rest unique letters at the remaining two positions in the 4-letter word.

(3) the number of all 4-letter distinguishable words having two repeating "S" is

$C_4^2 \cdot 9 \cdot 8 = \left(\frac{4 \cdot 3}{2}\right) \cdot 9 \cdot 8$ = 6*9*8 = 432.

Here $C_4^2 = \frac{4 \cdot 3}{2}$ = 6 is the number ow ways to select two positions for S in the 4-letter word,without looking the order. Next, the factors 9 and 8 are to calculate the number of placing the rest unique letters at the remaining two positions in the 4-letter word.

(4) the number of 4-letter distinguishable words having two repeating "E" and two repeating "S" is

$\frac{4!}{2! \cdot 2!} = \frac{24}{4}$ = 6.

Finally, I sum up the found numbers 5040 + 432 + 432 + 6 = 5910.

<u>ANSWER</u>. There are 5910 such distinguishable 4-letter words.

Example 25

Two persons A and B are to draw alternately one ball at a time from an urn containing 3 white and 2 black balls,
drawn balls not being replaced. If A takes the first turn, what is the probability that A will be the first to draw white ?
Solution
We may think about the space of events as all possible sequences of the letters W and B of the length 5.

There are 120 permutations of 5 items; but if we consider the distinguishable arrangements of these sequences, we have only $\frac{120}{3!\cdot 2!} = \frac{120}{6\cdot 2} = \frac{120}{12}$ = 10 distinguishable orderings, so the space of events has 10 elements.

Next, the "favorable" arrangements are those that EITHER start from W (and then A takes white ball first),

OR those that start from BBW (and then there is only one such sequence BBWWW, where, again, A takes white ball first).

The number of distinguishable sequences W _ _ _ _ is $\frac{4!}{2!\cdot 2!} = \frac{24}{4}$ = 6.

The sequence BBWWW is a unique of that kind.

So, the number of favorable distinguishable sequences is (6+1) = 7, and the total space of events has 10 elements.

THEREFORE, the probability under the problem's question is P = $\frac{\text{favorable}}{\text{total}} = \frac{7}{10}$ = 0.7 = 70%.

3.7 Combination

In mathematics, a **combination** is a way of selecting items from a collection where the order of selection does not matter. Suppose we have a set of three numbers P, Q and R. Then in how many ways we can select two numbers from each set, is defined by combination.

Definition of Combination in Math

The combination is defined as “An arrangement of objects where the order in which the objects are selected does not matter.” The combination means “Selection of things”, where the order of things has no importance.

For example,

if we want to buy a milkshake and we are allowed to combine any 3 flavours from Apple, Banana, Cherry, and Durian, then the combination of Apple, Banana, and Cherry is the same as the combination Banana, Apple, Cherry.

So if we are supposed to make a combination out of these possible flavours, then firstly, let us shorten the name of the fruits by selecting the first letter of their names.

We only have 4 possible combinations for the question above ABC, ABD, ACD, and BCD. Also, do notice that these are the only possible combination. This can be easily understood by the combination Formula.

Combination Formula

The Combination of 4 objects taken 3 at a time are the same as the number of subgroups of 3 objects taken from 4 objects. Take another example, given three fruits; say an apple, an orange, and a pear, three combinations of two can be drawn from this set: an apple and a pear; an apple and an orange; or a pear and an orange. More formally, a *k*-combination of a set is a subset of *k* distinct elements of *S*. If the set has *n* elements, the number of *k*-combinations is equal to the binomial coefficient.

$\mathbf{{}^nC_k = [(n)(n-1)(n-2)....(n-k+1)]/[(k-1)(k-2).......(1)]}$

which can be written as;

$\mathbf{{}^nC_k = n!/k!(n-k)!}$, **when n>k**

$\mathbf{{}^nC_k = 0}$, **when n<k**

Where n = distinct object to choose from

C = Combination

K = spaces to fill (Where k can be replaced by r also)

The combination can also be represented as: –$\mathbf{{}^nC_r}$, $\mathbf{{}_nC_r}$, **C(n,r)**, $\mathbf{C_r^n}$

Relation between Permutation and Combination

The combination is a type of permutation where the order of the selection is not considered. Hence, the count of permutation is always more than the number of the combination. This is the basic difference between permutation and combination. Now let us find out how these two are related.

Theorem : $\mathbf{{}^nP_r} = \mathbf{{}^nC_r}.r!$

Corresponding to each combination of ${}^nC_r$, we have r! permutations because r objects in every combination can be rearranged in r! ways.

Proof:

${}^nP_r = {}^nC_r.r!$

$= [n!/r!(n-r)!].r!$

$= n!/(n-r)!$

Hence the theorem states true.

Theorem: $^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

Proof:

$nCr+nCr-1=n!!(n-r)!+n!(r-1)!(n-r+1)!$

$=n!(r-1)!(n-r)!+n!(r-1)!(n-r+1)(n-r)!$

$=n!(r-1)!(n-r)![1r+1(n-r+1)]$

$=n!(r-1)!(n-r)![n+1r(n-r+1)]$

$=(n+1)!(r)!(n+1-r)!$

$= {}^{n+1}C_{r}$

Example 26

A group of 3 lawn tennis players S, T, U. A team consisting of 2 players is to be formed. In how many ways can we do so?

Solution- In a combination problem, we know that the order of arrangement or selection does not matter.

Thus ST= TS, TU = UT, and SU=US.

Thus we have 3 ways of team selection.

By combination formula we have-

$^{3}C_{2}$ = 3!/2! (3-2)!

= (3.2.1)/(2.1.1) =3

Example 27

Find the number of subsets of the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} having 3 elements.

Solution: The set given here have 10 elements. We need to form subsets of 3 elements in any order. If we select {1,2,3} as first subset then it is same as {3,2,1}. Hence, we will use the formula of combination here.

Therefore, the number of subsets having 3 elements = $^{10}C_{3}$

= 10!/(10-3)!3!

= 10.9.87!/7!.3!

= 10.9.8/3.2

= 120 ways.

Combinations with Repetitions Formula

You can use the formula below to find out the number of combinations when repetition is allowed.

$$C(n,r) = (r + n - 1)!_{\overline{r!(n-1)!}}$$

Here, n = total number of elements in a set

r = number of elements that can be selected from a set

Example 28

There are five colored balls in a pool. All balls are of different colors. In how many ways can we choose four pool balls?

Solution

The order in which the balls can be selected does not matter in this case. The selection of balls can be repeated.

Total number of balls in the pool= n = 5

The number of balls to be selected = r = 4

Use the following formula to get the number of arrangements in which the four pool balls can be chosen.

$$C(n,r) = (r + n - 1)!_{\overline{r!(n-1)!}}$$

Substitute these values in the above formula:

$$C(5,4) = (4 + 5 - 1)!_{\overline{4!(5-1)!}}$$

$$C(5,4) = 8!_{\overline{4!4!}}$$

$$C(5,4) = 8!_{\overline{4!4!}}$$

$$C(5,4) = 40320_{\overline{576}}$$

$$C(5,4) = 70$$

Hence, the pool balls can be selected in 70 different ways.

Example 29

There are eight different ice-cream flavors in the ice-cream shop. In how many ways can we choose five flavors out of these eight flavors?

Solution

The order in which the flavors can be selected does not matter in this case. One ice-cream flavor can be selected multiple times.

Total number of ice-cream flavors = n = 8

The number of ice-cream flavors to be selected = r = 5

Use the following formula to get the number of arrangements in which the five ice-cream flavors can be chosen.

$$C(n,r) = (r + n - 1)!\frac{}{r!(n-1)!}$$

Substitute these values in the above formula:

$$C(8,5) = (5 + 8 - 1)!\frac{}{5!(8-1)!}$$

$$C(8,5) = 12!\frac{}{5!7!}$$

$$C(8,5) = 12!\frac{}{5!7!}$$

$$C(5,4) = 792$$

Hence, the ice-cream flavors can be selected in 792 ways.

Example 30

Harry has six different colored shirts. In how many ways can he hang the four shirts in the cupboard?

Solution

The order in which the shirts can be selected does not matter in this case. The shirts can be repeated.

Total number of shirts = n = 6

The number of shirts to be selected = r = 4

Use the following formula to get the number of arrangements in which the four shirts can be chosen.

$$C(n,r) = \frac{(r + n - 1)!}{r!(n-1)!}$$

Substitute these values in the above formula:

$$C(6,4) = \frac{(4 + 6 - 1)!}{4!(6-1)!}$$

$$C(6,4) = \frac{9!}{4!5!}$$

$$C(6,4) = \frac{362880}{2880}$$

$$C(6,4) = 126$$

Hence, the shirts can be displayed in 126 different ways.

Example 31

Alice has seven different chocolates. How many ways can five chocolates be selected?

Solution

The order in which the chocolates can be selected does not matter in this case. The flavors can be repeated.

Total number of chocolates = n = 7

The number of chocolates to be selected = r = 5

Use the following formula to get the number of arrangements in which the four shirts can be chosen.

$$C(n,r) = \frac{(r + n - 1)!}{r!(n-1)!}$$

Substitute these values in the above formula:

$$C(7,5) = \frac{(5 + 7 - 1)!}{5!(7-1)!}$$

$$C(7,5) = \frac{11!}{5!6!}$$

$$C(7,5) = \frac{362880}{2880}$$

$$C(7,5) = 462$$

Hence, the chocolates can be selected in 462 ways.

Example 31

Sam has five colored pencils. In how many ways can he select three pencils?

Solution

The order in which the pencils can be selected does not matter in this case. The pencils can be repeated.

Total number of pencils = n = 5

The number of pencils to be selected = r = 3

Use the following formula to get the number of arrangements in which the five pencils can be chosen.

$$C(n,r) = \frac{(r + n - 1)!}{r!(n-1)!}$$

Substitute these values in the above formula:

$$C(5,3) = \frac{(3 + 5 - 1)!}{3!(5-1)!}$$

$$C(5,3) = \frac{7!}{3!4!}$$

$$C(5,3) = \frac{5040}{144}$$

$$C(5,3) = 35$$

Hence, the pencils can be selected in 35 different ways.

Example 32

Mariah has ten different candies. How many ways can six candies be selected?

Solution

The order in which the candies can be selected does not matter in this case. The candies can be repeated.

Total number of candies = n = 10

The number of candies to be selected = r = 6

Use the following formula to get the number of arrangements in which the six candies can be chosen.

$$C(n,r) = \frac{(r + n - 1)!}{r!(n-1)!}$$

Substitute these values in the above formula:

$$C(10,6) = \frac{(6 + 10 - 1)!}{6!(10-1)!}$$

$$C(10,6) = \frac{15!}{6!9!}$$

$$C(10,6) = 5005$$

Hence, the candies can be selected in 5005 different ways.

Difference Between Permutation and Combination

The major difference between the permutation and combination are given below:

Permutation	Combination
Permutation means the selection of objects, where the order of selection matters	The combination means the selection of objects, in which the order of selection does not matter.
In other words, it is the arrangement of r objects taken out of n objects.	In other words, it is the selection of r objects taken out of n objects irrespective of the object arrangement.
The formula for permutation is ${}^nP_r = n! /(n-r)!$	The formula for combination is ${}^nC_r = n!/[r!(n-r)!]$

Self-assessment questions

1. How to solve recurrence relation of order two? Explain

2. Analyze solving recurrence relation by iteration

3. Differentiate permutation and combination

4. Explain combination with repetition with example

5. Explain permutation of indistinguishable objects with example

6. How to solve nonlinear homogeneous recurrence relation? Explain

7. How to solve recurrence relation by iteration? Explain

Let us sum up

Recurrence relations are equations that define sequences based on previous terms. They are fundamental in various fields such as computer science, mathematics, and engineering. A recurrence relation expresses a term of a sequence as a function of earlier terms. There are two main types: linear, where terms are linearly combined, and non-linear, where terms may be multiplied or involve more complex operations. Recurrence relations can be homogeneous, involving only previous terms, or non-homogeneous, including additional non-sequence-dependent terms. Solving them often involves methods like finding characteristic equations for linear recurrences, using generating functions, or identifying particular solutions for non-homogeneous cases. They are particularly useful in algorithm analysis, mathematical modeling, and dynamic programming, helping to understand recursive processes and optimize problem-solving strategies

Check your progress

1.Consider the recurrence relation $a_1=4$, $a_n=5n+a_{n-1}$. The value of a_{64} is _________
a) 10399
b) 23760
c) 75100
d) 53700

2. Determine the solution of the recurrence relation $F_n=20F_{n-1} - 25F_{n-2}$ where $F_0=4$ and $F_1=14$.
a) $a_n = 14*5^{n-1}$
b) $a_n = 7/2*2^n-1/2*6^n$
c) $a_n = 7/2*2^n-3/4*6^{n+1}$
d) $a_n = 3*2^n-1/2*3^n$

3. What is the recurrence relation for 1, 7, 31, 127, 499?
a) $b_{n+1}=5b_{n-1}+3$
b) $b_n=4b_n+7!$
c) $b_n=4b_{n-1}+3$
d) $b_n=b_{n-1}+1$

4. If $S_n=4S_{n-1}+12n$, where $S_0=6$ and $S_1=7$, find the solution for the recurrence relation.
a) $a_n=7(2^n)-29/6n6^n$
b) $a_n=6(6^n)+6/7n6^n$
c) $a_n=6(3^{n+1})-5n$
d) $a_n=nn-2/6n6^n$

5. Find the value of a_4 for the recurrence relation $a_n=2a_{n-1}+3$, with $a_0=6$.
a) 320
b) 221
c) 141
d) 65

6. The solution to the recurrence relation $a_n=a_{n-1}+2n$, with initial term $a_0=2$ are __________
a) 4n+7
b) 2(1+n)
c) $3n^2$
d) 5*(n+1)/2

7. Determine the solution for the recurrence relation $b_n=8b_{n-1}-12b_{n-2}$ with $b_0=3$ and $b_1=4$.
a) $7/2*2^n-1/2*6^n$
b) $2/3*7^n-5*4^n$
c) $4!*6^n$
d) $2/8^n$

8. What is the solution to the recurrence relation $a_n=5a_{n-1}+6a_{n-2}$?
a) $2n^2$
b) 6n
c) (3/2)n
d) n!*3

9. Determine the value of a2 for the recurrence relation $a_n = 17a_{n-1} + 30n$ with $a_0=3$.
a) 4387
b) 5484
c) 238
d) 1437

10. Determine the solution for the recurrence relation $a_n = 6a_{n-1}-8a_{n-2}$ provided initial conditions $a_0=3$ and $a_1=5$.
a) $a_n = 4 * 2^n – 3^n$
b) $a_n = 3 * 7^n – 5*3^n$
c) $a_n = 5 * 7^n$
d) $a_n = 3! * 5^n$

Glossary

1. Recurrence Relation**:** An equation that recursively defines a sequence or multidimensional array of values. Each term is defined as a function of preceding terms.

2. Initial Conditions**:** The starting values required to uniquely determine the solution of a recurrence relation.

3. Homogeneous Recurrence Relation**:** A recurrence relation in which each term is a linear combination of previous terms, and there is no additional non-homogeneous term

4. Non-Homogeneous Recurrence Relation**:** A recurrence relation that includes terms that are not solely linear combinations of preceding terms.

5. Characteristic Equation**:** An algebraic equation obtained from a homogeneous recurrence relation by substituting. The roots of this equation help find the general solution of the recurrence relation.

6. Solution of a Recurrence Relation**:** The expression or formula that provides the terms of the sequence or array defined by the recurrence relation.

7. Closed-Form Solution**:** An explicit formula for the n-th term of the sequence that does not depend on previous terms.

8. Linear Recurrence Relation**:** A recurrence relation where each term is a linear combination of previous terms.

9. Order of Recurrence Relation**:** The number of previous terms the relation depends on.

10. Particular Solution**:** A solution to a non-homogeneous recurrence relation that solves the non-homogeneous part. For example, if an=3an−1+4a_n = 3a_{n-1} + 4an=3an−1+4, a particular solution might be a constant.

11. General Solution**:** The solution to a homogeneous recurrence relation plus a particular solution to the non-homogeneous part.

12. Method of Generating Functions**:** A technique to solve recurrence relations by transforming them into algebraic equations using generating functions, which are formal power series.

13. Iterative Method**:** A technique where terms are computed by iterating the recurrence relation from initial conditions rather than solving the relation directly.

14. Matrix Method**:** A technique involving matrix algebra to solve linear recurrence relations by representing the recurrence as a matrix multiplication problem.

15. Z-Transform**:** A technique used primarily in discrete signal processing to analyze and solve linear recurrence relations by transforming them into algebraic equations.

16. Difference Equation**:** A discrete analog to differential equations, describing the relationship between values of a sequence. Recurrence relations are a type of difference equation.

17. Bounded Solution**:** A solution where the terms of the sequence are constrained within some finite limits.

18. Asymptotic Behavior**:** The behavior of a sequence as nnn approaches infinity. It often involves analyzing the growth rate of the sequence's terms.

Unit summary

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of previous terms. It provides a way to express sequences in terms of their preceding elements**.**

Common Techniques**:**

- Iteration Method**:** Compute a few terms manually to discern a pattern or formula.
- Generating Functions**:** Use generating functions to transform recurrence relations into algebraic equations that are easier to solve.
- Matrix Methods**:** Apply linear algebra techniques to solve linear recurrence relations, particularly for higher-order relations.

Suggested readings

The Concrete Tetrahedron by M. Kauers and P. Paule

Unit-IV

Unit-IV

4.1 Special types of matrices:

There are several special types of matrices available. Those are:

Square Matrix: A matrix is square if it has the same number of rows and columns.

Symmetric Matrix: A square matrix that is equal to its transpose. In other words, $A = A^T$.

Diagonal Matrix: A square matrix in which all elements outside the main diagonal are zero.

Identity Matrix: A diagonal matrix with all diagonal elements equal to 1.

Zero Matrix: A matrix in which all elements are zero.

Orthogonal Matrix: A square matrix whose rows and columns are orthogonal unit vectors (i.e., perpendicular and of unit length, $AA^T = 1$).

Triangular Matrix: A matrix in which all elements above or below the main diagonal are zero.

Sparse Matrix: A matrix in which most of the elements are zero.

Hermitian Matrix: A square matrix that is equal to its conjugate transpose.

Skew-Symmetric Matrix: A square matrix whose transpose equals its negative, i.e., $A = -A^T$

Toeplitz Matrix: A matrix in which each descending diagonal from left to right is constant.

Hankel Matrix: A matrix in which each ascending diagonal from left to right is constant.

Positive Definite Matrix: A symmetric matrix in which all eigenvalues are positive.

Positive Semidefinite Matrix: Similar to positive definite matrix but allows zero eigenvalues.

Toeplitz-Plus-Hankel Matrix: A matrix that is both Toeplitz and Hankel.

Stochastic Matrix: A square matrix whose rows each sum to 1.

Markov Matrix: A stochastic matrix used in probability theory to describe transitions between states in a Markov chain.

These are just a few examples, and there are many other types of matrices studied in various branches of mathematics and its applications.

Example of Square matrices:

1. 2x2 Matrix:

$$\begin{bmatrix} 2 & 7 \\ 5 & 6 \end{bmatrix}$$

2. 3x3 Matrix:

$$\begin{bmatrix} 5 & 2 & 0 \\ 4 & 8 & 3 \\ 3 & 8 & 1 \end{bmatrix}$$

Example of Symmetric matrices:

1. If the matrix A = $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 7 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 2 & 4 & 0 & 4 \end{bmatrix}$ then, $A^T = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 7 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 2 & 4 & 0 & 4 \end{bmatrix}$

Here, $A = A^T$, so the matrix A is symmetric.

2. If the matrix I = $\begin{bmatrix} 3 & 10 & 9 \\ 10 & -3 & 0 \\ 9 & 0 & 1 \end{bmatrix}$ then, I^T = $\begin{bmatrix} 3 & 10 & 9 \\ 10 & -3 & 0 \\ 9 & 0 & 1 \end{bmatrix}$

Here, $I = I^T$, so the matrix I is symmetric.

Example of Orthogonal matrices:

1. If the matrix A = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Here, AA^T = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ = I

Therefore, the matrix A is an orthogonal matrix.

2. If the matrix A = $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ then A^T = $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Here, AA^T= $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Therefore, the matrix A is an orthogonal matrix.

Example of Triangular matrices:

1. The matrix A = $\begin{bmatrix} 1 & 8 & 6 & 5 \\ 0 & 51 & 4 & 7 \\ 0 & 0 & 1 & 90 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ is an upper triangular matrix.
2. The matrix A = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 8 & 3 & 0 & 0 \\ 12 & 43 & 7 & 0 \\ 1 & 9 & 6 & 1 \end{bmatrix}$ is a lower triangular matrix.

4.2 Determinants

Every square matrix A of order n, we associate a determinant of order n, which is denoted by A. The determinant has a value and this value is real if the matrix A is real and may be real or complex, if the matrix A is complex. A determinant of order n is defined as,

$$|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix} =$$

$$\begin{bmatrix} (-1)^{1+1}a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \\ +(-1)^{1+2}a_{12} \begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \\ +(-1)^{1+3}a_{13} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\ + \cdots + (-1)^{1+n}a_{1n} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2(n-1)} \\ a_{31} & a_{32} & \cdots & a_{3(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n(n-1)} \end{vmatrix} \end{bmatrix}$$

Example 1**:**

Find the determinant of the following matrix $A = \begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix}$$

$= (-1)^{1+1}\, 5|1| + (-1)^{1+2}\, 3|4|$

$= 5 - 12$

$= -7$

Example 2:

Find the determinant of the following matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

$= (-1)^{1+1}\, 1 \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} + (-1)^{1+2}\, 2 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + (-1)^{1+3}\, 3 \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix}$

$= -3 + 2 + 6$

$= 5$

4.3 Properties of Determinants:

1. If all the elements of a row (or column) are zero then the value of the determinant is zero.
2. The determinants of a matrix and its transpose have same value i.e, $A = |A|^T$.
3. If any two rows (or columns) are interchanged then the value of the determinant is multiplied by −1.
4. If the corresponding elements of two rows (or columns) are proportional to each other, then the value of the determinant is zero.

5. If each element of a row (or column) is multiplied by a scalar k then the value of the determinant is multiplied by the scalar k. Therefore, if α is a factor of each element of a row (or column), then this factor α can be taken out of the determinant.

6. If a non-zero constant multiple of the elements of some row (or column) are added to the corresponding elements of some other row (or column) then the value of the determinant remains unchanged.

7. In general $A + B \neq A + B$.

8. $|\alpha A| = \alpha^n |A|$, where α is any scalar and A is a matrix of order n.

9. If A and B are two square matrices of the same order then $|AB| = |A||B|$.

4.4 Inverse of A Square Matrix:

A square non-singular matrix A of order n is said to be invertible, if there exists a non-singular square matrix B of order n such that

$AB = BA = I$

where, I is an identity matrix of order n. The matrix B is called inverse matrix of A and we write $B = A{-}1$ or $A = B\ {-}1$. Hence, we say that $A{-}1$ is the inverse of the matrix A if

$A{-}1A = AA{-}1 = I$

The inverse, $A{-}1$ of the matrix A is given by,

$$A^{-1} = \frac{adj\ A}{|A|}\ , |A|\ \neq 0$$

Where $adjA$ =adjoin matrix of A.

= transpose of the matrix of cofactors of A.

Example 3**:**

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$

Solution:

Now $A = \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix}$

= (1)(5) – (1)(4)

= 1 ≠ 0, so the given matrix is non-singular i.e. $A{-1}$ exists.

Now, we find the cofactors of elements of A.

C_{11} = M_{11} = 5

C_{12} = $-M_{12}$ = −1

C_{21} = $-M_{21}$ = −4

C_{22} = M_{22} = 1

Now, the matrix of cofactors C = $\begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} = \begin{vmatrix} 5 & -1 \\ -4 & 1 \end{vmatrix}$

So $adjA$ = C^T = $\begin{vmatrix} 5 & -1 \\ -4 & 1 \end{vmatrix}$ where C^T denotes transpose of the matrix C.

Now, using (1) we have A^{-1} = $\frac{1}{1}\begin{vmatrix} 5 & -4 \\ -1 & 1 \end{vmatrix} = \begin{vmatrix} 5 & -4 \\ -1 & 1 \end{vmatrix}$

Example 4:
Solution:

Find the inverse of the matrix $A = \begin{vmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix}$

We know that, $A^{-1} = \frac{adj\ A}{|A|}$, $|A| \neq 0$

Now, $A = \begin{vmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix}$

$$= 2\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= 2(1-0) - 2(-2-3) + 0(0-3)$$

= 12 ≠ 0

So the given matrix is non-singular i.e. A^{-1} exists.

Now, we find the cofactors of elements of A.

$$C_{11} = M_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$C_{12} = -M_{12} = -\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = 5$$

$$C_{13} = M_{13} = \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$C_{21} = -M_{21} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2$$

$$C_{22} = M_{22} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2$$

$$C_{23} = -M_{23} = -\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} = 6$$

$$C_{31} = M_{31} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2$$

$$C_{32} = -M_{32} = -\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = -2$$

$$C_{33} = M_{33} = \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = 6$$

Now, the matrix of cofactors $C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \begin{vmatrix} 1 & 5 & -3 \\ -2 & 2 & 6 \\ 2 & -2 & 6 \end{vmatrix}$

So, $adjA = C^T = \begin{vmatrix} 1 & -2 & 2 \\ 5 & 2 & -2 \\ -3 & 6 & 6 \end{vmatrix}$

Where, C^T denotes the transpose of the matrix C.

Now, using (1) we have $A^{-1} = \frac{1}{12}\begin{vmatrix} 1 & -2 & 2 \\ 5 & 2 & -2 \\ -3 & 6 & 6 \end{vmatrix}$

Example 5:

Solution:

Find the inverse of the matrix $A = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix}$

We know that, $A^{-1} = \frac{adj\,A}{|A|}$, $|A| \neq 0$

Now, $A = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix}$

$$= 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= 1(1-0) - 2(1-3) + 0(0-3) \qquad = 5 \neq 0$$

So the given matrix is non-singular i.e. A^{-1} exists.

Now, we find the cofactors of elements of A.

$$C_{11} = M_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$C_{12} = -M_{12} = -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = 2$$

$$C_{13} = M_{13} = \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$C_{21} = -M_{21} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2$$

$$C_{22} = M_{22} = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1$$

$$C_{23} = -M_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} = 6$$

$$C_{31} = M_{31} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2$$

$$C_{32} = -M_{32} = -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

$$C_{33} = M_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

Now, the matrix of cofactors C = $\begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$ = $\begin{vmatrix} 1 & 2 & -3 \\ -2 & 1 & 6 \\ 2 & -1 & -1 \end{vmatrix}$

So, $adjA$ = C^T = $\begin{vmatrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ -3 & 6 & -1 \end{vmatrix}$

Where, C^T denotes the transpose of the matrix C.

Now, using (1) we have $A^{-1} = \frac{1}{5} \begin{vmatrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ -3 & 6 & -1 \end{vmatrix}$

4.5 Cramer's Rule for Solving the Linear Equations

Consider the following system of n linear non-homogeneous equations in n variables (unknowns) as,

$$\left.\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & \cdots & + & a_{1n}x_n & = b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & \cdots & + & a_{2n}x_n & = b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & \cdots & + & a_{3n}x_n & = b_3 \\
\cdots\ \cdots & & \cdots\ \cdots & & \cdots\ \cdots & & \cdots & \cdots & & \cdots\ \cdots & \cdots \\
\cdots\ \cdots & & \cdots\ \cdots & & \cdots\ \cdots & & \cdots & \cdots & & \cdots\ \cdots & \cdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \cdots & \cdots & + & a_{nn} & = b_n
\end{array}\right\} \qquad (1)$$

The system (1) can be written in matrix form as,

$$AX = B \qquad (2)$$

Where, $A = \begin{vmatrix}
a_{11} & + & a_{12} & + & a_{13} & + & \cdots & \cdots & \cdots & a_{1n} \\
a_{21} & + & a_{22} & + & a_{23} & + & \cdots & \cdots & \cdots & a_{2n} \\
a_{31} & + & a_{32} & + & a_{33} & + & \cdots & \cdots & \cdots & a_{3n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n1} & + & a_{n2} & + & a_{n3} & + & \cdots & \cdots & \cdots & a_{nn}
\end{vmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$

Let A be non-singular matrix. Then the solution of system (2) by Cramer's rule is given by,

$$x_i = \frac{|A_i|}{|A|}, i = 1, 2, 3, .. n. \qquad (3)$$

Where, $|A_i|$ is the determinant of the matrix Ai obtained by replacing the i-th column of matrix A by the right-hand side column vector B. Now, there are three possibilities.

1. If $A \neq 0$ then the system (1) is consistent and has a unique solution which is given by (3).

2. If $A = 0$ and at least one Ai is not equal to zero then the system (1) is inconsistent i.e., it has no solution.

3. If $A = 0$ and all Ai are equal to zero then the system (1) is consistent and has infinite number of solutions.

Example 6:
Solve the following system of non-homogeneous equations using Cramer's rule.

$$\left.\begin{aligned} x - y + z &= 4 \\ 2x + y - 3z &= 0 \\ x + y + z &= 2 \end{aligned}\right\}$$

Solution:

The given system of non-homogeneous equations is,

$$\left.\begin{aligned} x - y + z &= 4 \\ 2x + y - 3z &= 0 \\ x + y + z &= 2 \end{aligned}\right\} \qquad (1)$$

The system (1) can be written in matrix form as

$AX = B$ (2)

Where, $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $and\ B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Here, $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(1+3) + 1(2+3) + 1(2-1)$

$= 10 \neq 0$

So, the given system of equations has a unique solution.

Now, $|A_1| = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix}$

$= 4(1+3) + 1(0+6) + 1(0-2)$

$= 20$

$|A_2| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix}$

$= 1(0+6) - 4(2+3) + 1(4-0)$

$= -10$

$|A_3| = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$

$$= 1(2+0) - 1(4-0) + 4(2-1)$$

$$= 10$$

Using Cramer's rule, the solution of system (1) is given by,

$$x = \frac{|A_1|}{|A|}, \qquad y = \frac{|A_2|}{|A|}, \qquad z = \frac{|A_3|}{|A|}$$

$$x = \frac{20}{10}, \qquad y = \frac{-10}{10}, \qquad z = \frac{10}{10}$$

$$x = 2, \qquad y = -1, \qquad z = 1$$

Example 7**:**
Using Cramer's rule, show that the following system of non-homogeneous equations has infinite number of solutions.

$$\left.\begin{aligned} x - y + 3z = 3 \\ 2x + 3y + z = 2 \\ x + 2y + 4z = 5 \end{aligned}\right\}$$

Solution:

The given system of non-homogeneous equations is,

$$\left.\begin{aligned} x - y + 3z = 3 \\ 2x + 3y + z = 2 \\ x + 2y + 4z = 5 \end{aligned}\right\} \qquad (1)$$

The system (1) can be written in matrix form as

$$AX = B \qquad (2)$$

Where, $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $and\ B = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

Here, $|A| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix}$

$$= 1(12-2) + 1(8-3) + 3(4-9)$$

$$= 10 + 5 - 15 = 0$$

Now, $|A_1| = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix}$

$= 3(12 - 2) + 1(8 - 5) + 3(4 - 15)$

$= 30 + 3 - 33 = 0$

$|A_2| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix}$

$= 1(8 - 5) - 3(8 - 3) + 3(10 - 6)$

$= 3 - 15 + 12 = 0$

$|A_3| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix}$

$= 1(15 - 4) + 1(10 - 6) + 3(4 - 9)$

$= 11 + 4 - 15 = 0$

Since $|A|$ = 0, $|A_1|$ = 0, $|A_2|$ = 0 and $|A_3|$ = 0 so by Cramer's rule the given system of equations is consistent and it has infinite number of solutions.

4.6 Elementary Operations

There are two types of elementary operations. (i) Elementary Row Operations, (ii) Elementary Column Operations.

Elementary Row Operations: The following three elementary row operations are applied on matrices.

a) Interchanging of two rows: $R_i \leftrightarrow R_j$
b) Multiplication of a row by a non-zero constant k: $R_i \rightarrow kR_i$
c) Addition of a non-zero constant multiple of one row to another row: $R_i \rightarrow R_i + kR_j$

Elementary Column Operations: The following three elementary column operations are applied on matrices.

1. Interchanging of two column: $C_i \leftrightarrow C_j$
2. Multiplication of a column by a non-zero constant k: $C_i \rightarrow kC_i$
3. Addition of a non-zero constant multiple of one column to another column: $C_i \rightarrow C_i + kC_j$

A matrix is said to be of rank r when (i) It has at least one non-zero minor of order r, and (ii) Every minor of order higher than r vanishes.

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix. The rank of a matrix A shall be denoted by (A).

4.7 Rank of a Matrix

Characteristics of a Rank Matrix are:

(i) Rank of a matrix A and its transpose AT are the same.
(ii) Rank of a null matrix is zero.
(iii) Rank of an identity matrix is equal to the order of that identity matrix.
(iv) Rank of a non-singular matrix is equal to the order of that matrix.
(v) If a matrix has a non-zero minor of order r then its rank is ≥ r.
(vi) If all the minors of order r + 1 of a matrix are zero then its rank is ≤ r.

Example 8**:**

Determine the rank of the following matrix.

$$A = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 2 & -2 & 6 & 4 \\ 3 & -3 & 9 & 6 \end{bmatrix}$$

Solution:

Here order of matrix A is 3 × 4. Therefore, the rank of matrix A must be ≤ 3. The square sub-matrices of order 3 × 3 of the given matrix are,

$$A_1 = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & 6 \\ 3 & -3 & 9 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} -1 & 3 & 2 \\ -2 & 6 & 4 \\ -3 & 9 & 6 \end{bmatrix}$$

Now, $|A_1| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & -2 & 6 \\ 3 & -3 & 9 \end{vmatrix}$

= 1 −18 + 18 + 1 18 − 18 + 3 −6 + 6 = 0

$$|A_2| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{vmatrix}$$

= 1 −12 + 12 + 1 12 − 12 + 2 −6 + 6 = 0

$$|A_3| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{vmatrix}$$

= 1 36 − 36 − 3 12 − 12 + 2 18 − 18 = 0

$$|A_4| = \begin{vmatrix} -1 & 3 & 2 \\ -2 & 6 & 4 \\ -3 & 9 & 6 \end{vmatrix}$$

= −1 36 − 36 − 3 −12 + 12 + 2 −18 + 18 = 0

Here determinants of all square sub-matrices of order 3 × 3 of the given matrix are zero so the rank of the given matrix is less than 3. Now all square sub- matrices of order 2 × 2 of the given matrix are,

$B_1 = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $B_3 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B_4 = \begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix}$,

$B_5 = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}$, $B_6 = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$, $B_7 = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$, $B_8 = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$,

$B_9 = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$, $B_{10} = \begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix}$, $B_{11} = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$, $B_{12} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$,

$B_{13} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$, $B_{14} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $B_{15} = \begin{bmatrix} 1 & 3 \\ -3 & 9 \end{bmatrix}$, $B_{16} = \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix}$,

$B_{17} = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$

Here it is clear that the determinants of all square sub-matrices of order 2 × 2 of the given matrix are zero so the rank of the given matrix is less than 2.

Now the given matrix has at least one square sub-matrices of order 1 × 1 namely C_1 = $[1]$ whose determinant is non-zero, so the rank of given matrix.

$$A = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 2 & -2 & 6 & 4 \\ 3 & -3 & 9 & 6 \end{bmatrix} \ is\ 1.$$

Example 9**:** Determine the rank of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Solution:

Here order of matrix A is 3 × 3. Therefore, the rank of matrix A must be ≤ 3.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

= 1 45 − 48 − 2 36 − 42 + 3 32 − 35
= −3 − 2 −6 + 3 −3 = 0

Therefore, the rank of matrix A must be < 3. Now the given matrix has at least one square sub-matrices of order 2 × 2 namely $A_1 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ whose determinant is non-zero, so the rank of given matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \; is \; 2.$$

4.8 Characteristic Roots and Characteristic Vectors

Let $A = \left[a_{ij}\right]$ be a given n × n matrix and consider the vector equation

$AX = \lambda X$ (1)

Here, X is an unknown vectors and λ an unknown scalars, and want to determine both.

Clearly, the zero vector X = 0 is a solution of (1) for any value of λ. A value of λ for which (1) has a solution $X \neq 0$ is called a characteristic value or Eigen value or latent root or proper value of the matrix A. The corresponding solutions $X \neq 0$ of (1) called characteristic vectors or Eigen vectors of A corresponding to that value λ.

Now, (1) can be written as,

$$\left.\begin{array}{lllllllllll}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & \cdots & + & a_{1n}x_n & = b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & \cdots & + & a_{2n}x_n & = b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & \cdots & + & a_{3n}x_n & = b_3 \\
\cdots\ \cdots & & \cdots\ \cdots & & \cdots\ \cdots & & \cdots & \cdots & & \cdots\ \cdots & \cdots \\
\cdots\ \cdots & & \cdots\ \cdots & & \cdots\ \cdots & & \cdots & \cdots & & \cdots\ \cdots & \cdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \cdots & \cdots & + & a_{nn}x_n & = b_n
\end{array}\right\}$$

OR

$$\left.\begin{array}{lllllllllll}
(a_{11}-\lambda) & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & \cdots & + & a_{1n}x_n & = 0 \\
a_{21}x_1 & + & (a_{22}-\lambda) & + & a_{23}x_3 & + & \cdots & \cdots & + & a_{2n}x_n & = 0 \\
a_{31}x_1 & + & a_{32}x_2 & + & (a_{33}-\lambda) & + & \cdots & \cdots & + & a_{3n}x_n & = 0 \\
\cdots\ \cdots & & \cdots\ \cdots & & \cdots\ \cdots & & \cdots & \cdots & & \cdots\ \cdots & \cdots \\
\cdots\ \cdots & & \cdots\ \cdots & & \cdots\ \cdots & & \cdots & \cdots & & \cdots\ \cdots\ \cdots & \cdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \cdots & \cdots & + & (a_{nn}-\lambda) & = 0
\end{array}\right\}$$

In matrix notation the above system of equations can be written as,
$(A - \lambda I)\ X = 0$ (2)

This homogeneous linear system of equations has a non-trivial solution if the corresponding determinant of the coefficients (i.e. $A - \lambda I$) is zero.

$$\text{So, } |A - \lambda I| = \begin{vmatrix} (a_{11}-\lambda) & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & (a_{33}-\lambda) & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & (a_{nn}-\lambda) \end{vmatrix} = 0$$

The equation (3) is called the characteristic equation of the matrix A. By developing $|A - \lambda I|$, we obtain a polynomial of n^{th} degree in λ. This is called the characteristic polynomial of A.

Example 10:
Find the Eigen values and Eigen vectors of the matrix.

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

Solution

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

or, $[(5-\lambda)(2-\lambda) - 4] = 0$

or, $(10 - 5\lambda - 2\lambda + \lambda^2 - 4) = 0$

or, $[\lambda^2 - 7\lambda + 6] = 0$

or, $[\lambda^2 - 6\lambda - \lambda + 6] = 0$

or, $\lambda(\lambda - 6) - 1(\lambda - 6) = 0$

or, $(\lambda - 6)\ (\lambda - 1) = 0$

or, $\lambda = 1, 6$

Thus $\lambda = 1, 6$ are the eigen values of A.

Now the Eigen Vector Corresponding to Eigen Value $\boldsymbol{\lambda} = \mathbf{1}$.

Consider $(A - \lambda I)\ X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

So, (A – 1I) X = 0

or, (A – 1) X = 0

or, $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or, $\left.\begin{matrix} 4x_1 & + & 4x_2 & = 0 \\ x_1 & + & x_2 & = 0 \end{matrix}\right\}$

For solving the above system of equations, let $x_2 = c_1 \neq 0$ (arbitrary constant).

Using $x_2 = c_1$ in (1), we have $x_1 = -c_1$

Hence, $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -C_1 \\ C_1 \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now the Eigen Vector Corresponding to Eigen Value $\boldsymbol{\lambda}$ = 6.

Consider $(A - \lambda I)\ X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

So, (A – 6I) X = 0

or, $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or, $\left.\begin{matrix} -x_1 & + & 4x_2 & = 0 \\ x_1 & - & 4x_2 & = 0 \end{matrix}\right\}$

For solving the above system of equations, let $x_2 = c_1 \neq 0$ (arbitrary constant).

Using $x_2 = c_2$ in (2), we have $x_1 = 4c_1$

Hence, $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4C_2 \\ C_2 \end{bmatrix} = C_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Example 11:
Find the Eigen values and Eigen vectors of the matrix.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:
The characteristic equation of the given matrix is,

$|A - \lambda I|$ = 0

or, $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix}$ = 0

or, (2 – λ)[(2 – λ)(1 – λ) – 0] – 1[1 − λ – 0] + 1[0 – 0] = 0

or, $(2 - \lambda)^2(1 - \lambda) - (1 - \lambda) = 0$

or, $(1 - \lambda)[(2 - \lambda)^2 - 1] = 0$

or, $(1 - \lambda)[4 + \lambda^2 - 4\lambda - 1] = 0$

or, $(1 - \lambda)[\lambda^2 - 4\lambda + 3] = 0$

or, $(1 - \lambda)(\lambda - 3)(\lambda - 1) = 0$

or, λ = 1, 1, 3

Thus λ = 1, 1, 3 are the eigen values of A.

Now, the Eigen Vector Corresponding To Eigen Value $\boldsymbol{\lambda} = \mathbf{1}$

Consider $(A - \lambda I)\ X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

So, $(A - 1I) = 0$

or $(A - I)X = 0$

or $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or $x_1 + x_2 + x_3 = 0$

Let $x_2 = k_1$, $x_3 = k_2$ so we get $x_1 = -(k_1 + k_2)$

Hence, X = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (k_1 & + & k_2) \\ & k_1 & \\ & k_2 & \end{bmatrix}$

Here there are two linear independent solutions given by,
(i). Taking $x_2 = k_1 = 0$ and $x_3 = k_2(arbitrary)$ we have

$$X_1 = \begin{bmatrix} -k_2 \\ 0 \\ k_2 \end{bmatrix} = k_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(ii). Taking $x_3 = k_2 = 0$ and $x_2 = k_1(arbitrary)$ we have

$$X_2 = \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} = k_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Now, the Eigen Vector Corresponding To Eigen Value $\boldsymbol{\lambda} = 3$

Consider $(A - \lambda I)\ X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

So, $(A - 3I)\ X = 0$

or, $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or, $-x1 + x2 + x3 = 0$ (1)

$x1 - x2 + x3 = 0$ (2)

$-2x3 = 0$ (3)

After solving (3), we have $x_3 = 0$. Putting $x_3 = 0$ in (1) and (2), we get,

$-x1 + x2 = 0$ (4)

$x1 - x2 = 0$ (5)

For solving (4) and (5), we let $x_2 = k \neq 0$ ($constant$) so we have $x_1 = k$.

Hence, $X = X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

4.9 Cayley–Hamilton Theorem

Every square matrix A satisfies its characteristic equation.

i.e., $A^n - c_1A^{n-1} + \cdots + (-1)^{-1}c_{n-1}A + (-1)^n c_n I = 0$.

Proof:

The cofactors of the elements of the determinant $|A - \lambda I|$ are polynomials in λ of degree (n – 1) or less. Therefore, the elements of the adjoint matrix (transpose of the cofactor matrix) are also polynomials in λ of degree (n – 1) or less. Hence, we can express the adjoint matrix as a polynomial in λ whose coefficients $B1$, $B1 \ldots Bn$ are square matrices of order n having elements as functions of the elements of the matrix A. Thus, we can write

$(A - \lambda I) = B_1\lambda^{n-1} + B_2\lambda^{n-2} + \cdots + B_{n-1}\lambda + B_n$

Since $(A - \lambda I)$ $(A - \lambda I) = |A - \lambda I|\, I$

$$\text{So, } (A - \lambda I)\begin{pmatrix} B_1\lambda^{n-1} + B_2\lambda^{n-2} \\ + \cdots + B_{n-1}\lambda + B_n \end{pmatrix} = \begin{bmatrix} \lambda^n - C_1\lambda^{n-1} \\ + \cdots + \\ +(-1)^{n-1}C_{n-1}\lambda \\ +(-1)^n C_n \end{bmatrix} I$$

Comparing the coefficients of various power of λ, we have,

$$\left.\begin{array}{c} B_1 = I \\ AB_1 - B_2 = -C_1 I \\ AB_2 - B_3 = C_2 I \\ \cdots \quad \cdots \quad \cdots \\ AB_{n-1} - B_n = (-1)^{n-1} C_{n-1}\lambda \\ AB_n = (-1)^n C_n I \end{array}\right\}$$

Pre-multiplying these equations by An, $An-1$, … … . . , A, I respectively and adding, we have

$A_n - c_1 A_{n-1} + \cdots + (-1)^{-1} c_{n-1} A + (-1)^n c_n I = 0$

which completes the proof.

Example 12:

Verify Cayley–Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and hence find the inverse of A.

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} = 0$

or, $1 - \lambda\ 3 - \lambda - 2 = 0$

or, $\lambda^2 - 4\lambda + 1 = 0$

According to Cayley–Hamilton theorem “Every square matrix satisfies its characteristic equation”, so $A^2 - 4A + I = O$

Verification:

L.H.S. = $A^2 - 4A + I = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

= $\begin{bmatrix} (1)(1) + (2)(1) & (1)(2) + (2)(3) \\ (1)(1) + (3)(1) & (1)(2) + (3)(3) \end{bmatrix}$

= $\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 4 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

= $\begin{bmatrix} 3 - 4 + 1 & 8 - 8 + 0 \\ 4 - 4 + 0 & 11 - 12 + 1 \end{bmatrix}$

= $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

= R.H.S.

Hence, Cayley–Hamilton theorem is verified.

Now pre-multiplying both sides of equation $A^2 - 4A + I = O$ by A^{-1}, we have

$A - 4I + A^{-1} = O$

or $A^{-1} = 4I - A$

or $A^{-1} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

= $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

= $\begin{bmatrix} 4 - 1 & 0 - 2 \\ 0 - 1 & 4 - 3 \end{bmatrix}$

= $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

Example 13:

Verify Cayley–Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the inverse of A.

Also, find the matrix represented by,

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$
Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

or $(2 - \lambda)(1 - \lambda)(2 - \lambda) - 1(0 - 0) + 1[0 - 1 - \lambda] = 0$

or $(1 - \lambda)[(2 - \lambda)(2 - \lambda) - 1] = 0$

or $(1 - \lambda)(\lambda^2 - 4\lambda + 3) = 0$

or $(\lambda^2 - 4\lambda + 3 - \lambda^3 + 4\lambda^2 - 3\lambda) = 0$

or $-(\lambda^3 - 5\lambda^2 + 7\lambda - 3) = 0$

or $(\lambda^3 - 5\lambda^2 + 7\lambda - 3) = 0$

According to Cayley–Hamilton theorem “Every square matrix satisfies its characteristic equation”, so

$A^3 - 5A^2 + 7A - 3I = O$

Verification:

We have $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$A^3 = A^2 A = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$

L.H.S. = $A^3 - 5A^2 + 7A - 3I$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - \begin{bmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{bmatrix} + \begin{bmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

=R.H.S.

Hence, Cayley–Hamilton theorem is verified.

Now pre-multiplying both sides of equation $A^3 - 5A^2 + 7A - 3I = O$ by A^{-1}, we have

$A^2 - 5A + 7I - 3A^{-1} = O$

or $3A{-}1 = A^2 - 5A + 7I$

or $3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

or $3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

or $3A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

So, $A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$= O + A^2 + A + I$ (Since $A^3 - 5A^2 + 7A - 3I = O$)

$= A^2 + A + I$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Self-assessment questions

1. Explain the Special types of matrices

2. Write Properties of Determinants

3. Analyze Inverse of A Square Matrix

4. Summarize Cramer's Rule for Solving the Linear Equations

5. Elucidate Elementary Operations of matrix

6. Explain Rank of a Matrix with example

7. Write about Characteristic Roots and Characteristic Vectors

8. Explain Cayley–Hamilton Theorem with example

Let us sum up

Matrices are rectangular arrays of numbers, symbols, or expressions arranged in rows and columns. They serve as a fundamental tool in various fields, including mathematics, physics, computer science, and engineering. Matrices can be used to represent and solve systems of linear equations, perform transformations, and model various real-world scenarios. Key operations involving matrices include addition, subtraction, multiplication, and finding the determinant or inverse. Addition and subtraction of matrices require that the matrices have the same dimensions, with corresponding elements combined or subtracted. Matrix multiplication involves the dot product of rows and columns, and it is crucial for tasks such as linear transformations and solving systems of linear equations. The inverse of a matrix, if it exists, allows for the solution of matrix equations and is pivotal in various applications, including computer graphics and data analysis. Overall, matrices provide a powerful framework for handling and manipulating complex mathematical problems efficiently.

Check your progress

1. What is the determinant of the matrix $\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$
   a) -1
   b) 1
   c) 2
   d) 7

2. If matrix A is 3 x 4 and matrix B is 4 x 2, what are the dimensions of the product AB?
   a) 3 X 2
   b) 4 X 4
   c) 3 X 4
   d) 4 X 2

3. what is the transpose of matrix A = $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

b) $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$

4. A is 3 × 4 matrix, if A^TB and BA^T are defined then, B is a _____ matrix.
   (a) 4 × 3
   (b) 3 × 3
   (c) 4 × 4
   (d) 3 × 4

5. The matrix which follows the conditions m=n is called?
   a) Square matrix
   b) Rectangular matrix
   c) Scalar matrix
   d) Diagonal matrix

6. Consider the matrix, $A = \begin{pmatrix} 4 & 6 & 9 \\ 12 & 11 & 10 \end{pmatrix}$. What is the type of matrix?
   a) Row matrix
   b) Column matrix
   c) Horizontal matrix
   d) Vertical matrix

7. If A and B are symmetric matrices of the same order, then
   a) AB is a symmetric matrix
   b) A – Bis askew-symmetric matrix
   c) AB + BA is a symmetric matrix
   d) AB – BA is a symmetric matrix

8. If A is a square matrix, then A – A' is a
   a) diagonal matrix
   b) skew-symmetric matrix
   c) symmetric matrix
   d) none of these

9. For a Non-trival solution |A| is
   a) |A| < 0
   b) |A| > 0
   c) |A| ≠ 0
   d) |A| = 0

10. if A(BC) = (AB)C, then with respect to multiplication this law is called,

a) Inverse law
b) Associative law
c) Cramers law
d) Additive law

Unit summary

Matrices are fundamental mathematical structures consisting of rectangular arrays of numbers arranged in rows and columns. They come in various types, including square matrices, which have the same number of rows and columns, diagonal matrices with non-diagonal elements being zero, identity matrices where the diagonal elements are all 1, and zero matrices where all elements are zero. Additionally, symmetric matrices are equal to their transpose. Matrix operations include addition and subtraction, where corresponding elements are combined; scalar multiplication, which involves multiplying every element by a scalar; matrix multiplication, which is more complex and involves summing products of rows and columns; and transposition, where rows and columns are swapped.

Determinants provide a scalar value for square matrices, and inverses are matrices that, when multiplied by the original matrix, yield the identity matrix. Special forms such as Row Echelon Form (REF) and Reduced Row Echelon Form (RREF) are used for simplifying matrices and solving systems of linear equations. Matrices find extensive applications in various fields, including solving systems of equations, computer graphics for transformations and image processing, economics for modeling and data analysis, and engineering for system modeling and simulations. Understanding matrices and their operations is crucial for a wide range of theoretical and practical applications.

Glossary

Matrix: A rectangular array of numbers or elements arranged in rows and columns.
Element: An individual number or entry in a matrix.
Row: A horizontal line of elements in a matrix.
Column: A vertical line of elements in a matrix.
Square Matrix: A matrix with the same number of rows and columns.
Diagonal Matrix: A square matrix where all non-diagonal elements are zero.
Identity Matrix: A square matrix with 1s on the diagonal and 0s elsewhere.
Zero Matrix: A matrix in which all elements are zero.
Symmetric Matrix: A square matrix that is equal to its transpose.
Transpose: The operation of swapping rows and columns in a matrix.
Determinant: A scalar value that can be computed from a square matrix, providing information about the matrix's properties, such as invertibility.
Inverse Matrix: A matrix that, when multiplied by the original matrix, results in the identity matrix. Not all matrices have inverses.

Matrix Multiplication: An operation where two matrices are multiplied to produce a third matrix, where the element in position (i,j)(i, j)(i,j) is the sum of the products of elements from the iii-th row of the first matrix and the jjj-th column of the second matrix.
Scalar Multiplication: The process of multiplying every element in a matrix by a scalar (a single number).
Addition/Subtraction: Operations involving the element-wise addition or subtraction of two matrices of the same dimensions.
Row Echelon Form (REF): A form of a matrix where all zero rows are at the bottom, and each leading entry of a non-zero row is to the right of the leading entry of the row above it.
Reduced Row Echelon Form (RREF): A further simplified form of REF where each leading entry is 1 and is the only non-zero entry in its column.
Rank: The maximum number of linearly independent rows or columns in a matrix, indicating the dimension of the row space or column space.
Eigenvalue: A scalar associated with a square matrix, representing a factor by which the eigenvector is scaled during matrix transformation.
Eigenvector: A non-zero vector that, when multiplied by a square matrix, yields a scalar multiple of itself, defined by the associated eigenvalue

Suggested readings

Linear Algebra and Its Applications" by Gilbert Strang
Matrix Analysis" by Roger A. Horn and Charles R. Johnson

UNIT-V

5.1 Definition

5.2 Types of Graph

5.3 Properties of Graph

5.4 Trees, Degree and Cycle of Graph

5.5 Connectivity

5.6 Connected graphs

5.7 Euler graph

5.8 Hamiltonian circuit and Hamiltonian path

5.9 Planar graph

5.10 Complete graph

5.11 Bipartite graph

5.12 Hypercube graph

5.13 Matrix representations of graphs

UNIT-V

5.1 Definition

Graph Theory is the study of points and lines. In Mathematics, it is a sub-field that deals with the study of graphs. It is a pictorial representation that represents the Mathematical truth. Graph theory is the study of relationship between the vertices (nodes) and edges (lines).

Formally, a graph is denoted as a pair G(V, E).

Where V represents the finite set vertices and E represents the finite set edges.

Therefore, we can say a graph includes non-empty set of vertices V and set of edges E.

Example

Suppose, a Graph G=(V,E), where

Vertices, V={a,b,c,d}

Edges, E={{a,b},{a,c},{b,c},{c,d}}

5.2 Types of Graph

The graphs are basically of two types, directed and undirected. It is best understood by the figure given below. The arrow in the figure indicates the direction.

Directed Graph

In graph theory, a directed graph is a graph made up of a set of vertices connected by edges, in which the edges have a direction associated with them.

Undirected Graph

The undirected graph is defined as a graph where the set of nodes are connected together, in which all the edges are bidirectional. Sometimes, this type of graph is known as the undirected network.

Other types of graphs

- Null Graph: A graph that does not have edges.
- Simple graph: A graph that is undirected and does not have any loops or multiple edges.
- Multigraph: A graph with multiple edges between the same set of vertices. It has loops formed.
- Connected graph: A graph where any two vertices are connected by a path.
- Disconnected graph: A graph where any two vertices or nodes are disconnected by a path.
- Cycle Graph: A graph that completes a cycle.
- Complete Graph: When each pair of vertices are connected by an edge then such graph is called a complete graph
- Planar graph: When no two edges of a graph intersect and are all the vertices and edges are drawn in a single plane, then such a graph is called a planar graph

5.3 Properties of Graph

- The starting point of the network is known as root.
- When the same types of nodes are connected to one another, then the graph is known as an assortative graph, else it is called a disassortative graph.
- A cycle graph is said to be a graph that has a single cycle.
- When all the pairs of nodes are connected by a single edge it forms a complete graph.
- A graph is said to be in symmetry when each pair of vertices or nodes are connected in the same direction or in the reverse direction.
- When a graph has a single graph, it is a path graph.

5.4 Trees, Degree and Cycle of Graph

There are certain terms that are used in graph representation such as Degree, Trees, Cycle, etc. Let us learn them in brief.

Trees: A tree in a graph is the connection between undirected networks which are having only one path between any two vertices. It was introduced by British mathematician Arthur Cayley in 1857. The graph trees have only straight lines between the nodes in any specific direction but do not have any cycles or loops. Therefore trees are the directed graph.

Degree: A degree in a graph is mentioned to be the number of edges connected to a vertex. It is denoted deg(v), where v is a vertex of the graph. So basically it the measure of the vertex.

Cycle: A cycle is a closed path in a graph that forms a loop. When the starting and ending point is the same in a graph that contains a set of vertices, then the cycle of the graph is formed. When there is no repetition of the vertex in a closed circuit, then the cycle is a simple cycle. The cycle graph is denoted by C_n.

- A cycle that has an even number of edges or vertices is called Even Cycle.
- A cycle that has an odd number of edges or vertices is called Odd Cycle.

5.5 Connectivity

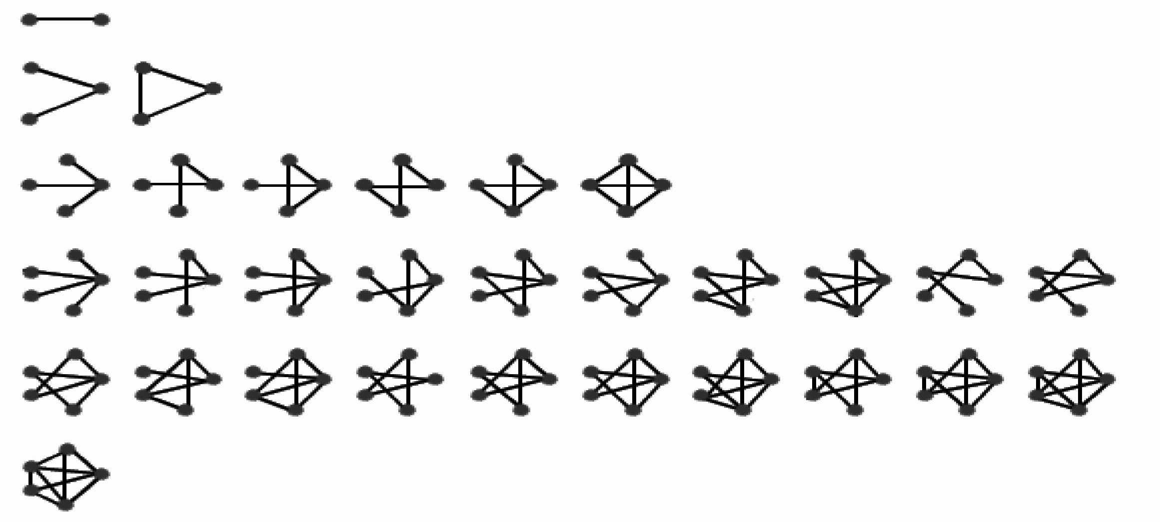
In Mathematics, the meaning of **connectivity** is one of the fundamental concepts of graph theory. It demands a minimum number of elements (nodes or edges) that require to be removed to isolate the remaining nodes into separated subgraphs. It is closely related to the principles of network flow problems. The connectivity of a graph is an essential measure of its flexibility as a network.

The word "**connectivity**" may belong to several applications in day to day life. Usually, it is referred to as the connection between two or more things or properties. In terms of different subjects, the definition of connectivity is described below:

1. In topology, the connected space is the topological space, which we cannot write in the form of union of two or more open non-empty subsets.
2. In the field of information and technology, the connectivity may relate to internet connectivity by which several individual computers, cell phones and LANs are connected to the global Internet.
3. In the field of transport planning, connectivity is also called permeability which relates to the limit to which urban structures limit the movement of vehicles and people.
4. The concept of pixel connectivity is used in the field of image processing.
5. Most importantly, in Mathematics, the term connectivity is utilized in graph theory. Graph connectivity is applicable in routing, network, network tolerance, transportation network, etc.

Connectivity Definition

Connectivity is one of the essential concepts in graph theory. A graph may be related to either **connected or disconnected** in terms of topological space. If there exists a path from one point in a graph to another point in the same graph, then it is called a connected graph. Else, it is called a disconnected graph. Below are the diagrams which show various types of connectivity in the graphs.



Connectivity in Graph Theory

A graph is a connected graph if, for each pair of vertices, there exists at least one single path which joins them. A connected graph may demand a minimum number of edges or vertices which are required to be removed to separate the other vertices from one another. The graph connectivity is the measure of the robustness of the graph as a network.

In a connected graph, if any of the vertices are removed, the graph gets disconnected. Then the graph is called a **vertex-connected graph**. On the other hand, when an edge is removed, the graph becomes disconnected. It is known as an **edge-connected graph.**

Properties of Connectivity

- The connected graph is called an undirected **graph,** which has at least one path between each pair of vertices.
- The graph that is connected by three vertices is called **1-vertex connected graph** since the removal of any of the vertices will lead to disconnection of the graph.
- If in a connected graph, the removal of one edge leads to the disconnection of the graph, such a graph is called **1-edge connected graph**.
- If there exists a set (say S) of edges (or vertices) in a connected graph, such that by removing all the edges of set S will result in a disconnected graph. Then the set S is called a **cut set**. If S consists of vertices, then it is called a **vertex-cut set**. Similarly, if it has edges, then it is called an **edge-cut set**.

- A bi-connected graph is a connected graph which has two vertices for which there are two disjoint paths between these two vertices.

5.6 **Connected Graphs**

There are different types of connected graphs explained in Maths. They are:

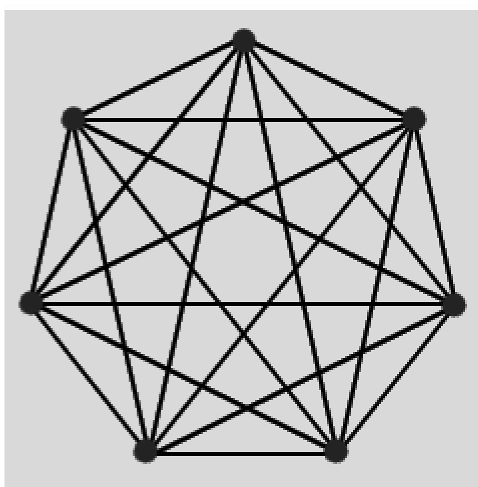
- Fully Connected Graph
- K-connected Graph
- Strongly Connected Graph

Let us learn them one by one.

Fully Connected Graph

In graph theory, the concept of a fully-connected graph is crucial. It is also termed as a complete graph. It is a connected graph where a unique edge connects each pair of vertices. In other words, for every two vertices of a whole or a fully connected graph, there is a distinct edge.

A fully connected graph is denoted by the symbol K_n, named after the great mathematician Kazimierz Kuratowski due to his contribution to graph theory. A complete graph K_n possesses $n/2(n-1)$ number of edges. Given below is a fully-connected or a complete graph containing 7 edges and is denoted by K_7.



K connected Graph

A graph is called a **k-connected graph** if it has the smallest set of k-vertices in such a way that if the set is removed, then the graph gets disconnected. Complete or fully-connected graphs do not come under this category because they don't get disconnected by removing any vertices. A set of graphs has a large number of k vertices based on which the graph is called k-vertex connected. It could be one-connected, two-connected or bi-connected, three-connected or tri-connected.

Strongly Connected Graph

A graph can be defined as a **strongly connected graph** if its every vertex can be reached from every other vertex in the graph. In other words, any directed graph is called **strongly connected** if there exists a path in each possible direction between each pair of vertices in the graph.

In a graph (say G) which may not be strongly connected itself, there may be a pair of vertices say (a and b) that are called strongly connected to each other if in case there exists a path in all the possible directions between a and b.

Examples of Connectivity

Example 1: If a complete graph has a total of 20 vertices, then find the number of edges it may contain.

Solution**:** The formula for the total number of edges in a k_{15} graph is given by;

Number of edges = n(n-1)/2

= 20(20-1)/2

=10(19)

=190

Hence, it contains 190 edges.

Example 2: If a graph has 40 edges, then how many vertices does it have?

Solution: As we know,

Number of edges = n (n-1)/2

40 = n(n-1)/2

n(n-1) = 80

$n^2 – n – 80 = 0$

On solving the above quadratic equation, we get;

$n \approx 9.45, -8.45$

Since, the number of vertices cannot be negative.

Therefore, $n \approx 9$

5.7 Euler Graph

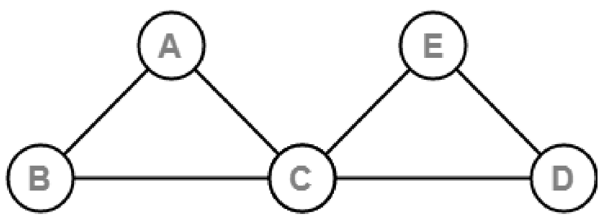
An Euler graph may be defined as-

Any connected graph is called as an Euler Graph if and only if all its vertices are of even degree.

OR An Euler Graph is a connected graph that contains an Euler Circuit.

Euler Graph Example-

The following graph is an example of an Euler graph-



Example of Euler Graph

Here,

- This graph is a connected graph and all its vertices are of even degree.
- Therefore, it is an Euler graph.

Euler Path-

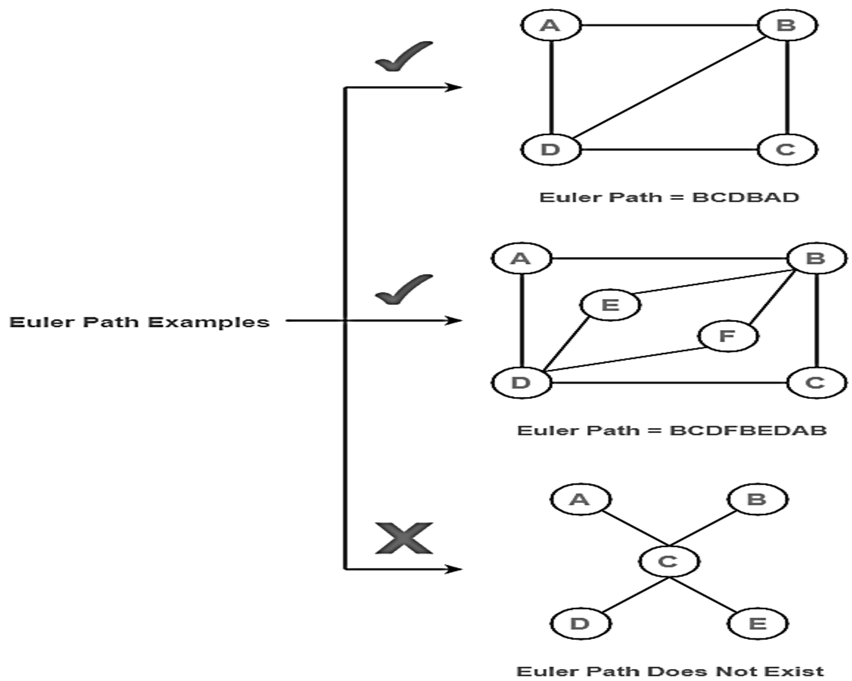
Euler path is also known as **Euler Trail** or **Euler Walk**.

If there exists a **Trail** in the connected graph that contains all the edges of the graph, then that trail is called as an Euler trail.

OR

If there exists a walk in the connected graph that visits every edge of the graph exactly once with or without repeating the vertices, then such a walk is called as an Euler walk.

Examples of Euler path are as follows



<u>**Euler Circuit-**</u>

Euler circuit is also known as **Euler Cycle** or **Euler Tour**

- If there exists a <u>**Circuit**</u> in the connected graph that contains all the edges of the graph, then that circuit is called as an Euler circuit.

OR

- If there exists a walk in the connected graph that starts and ends at the same vertex and visits every edge of the graph exactly once with or without repeating the vertices, then such a walk is called as an Euler circuit.

OR

- An Euler trail that starts and ends at the same vertex is called as an Euler circuit.

OR

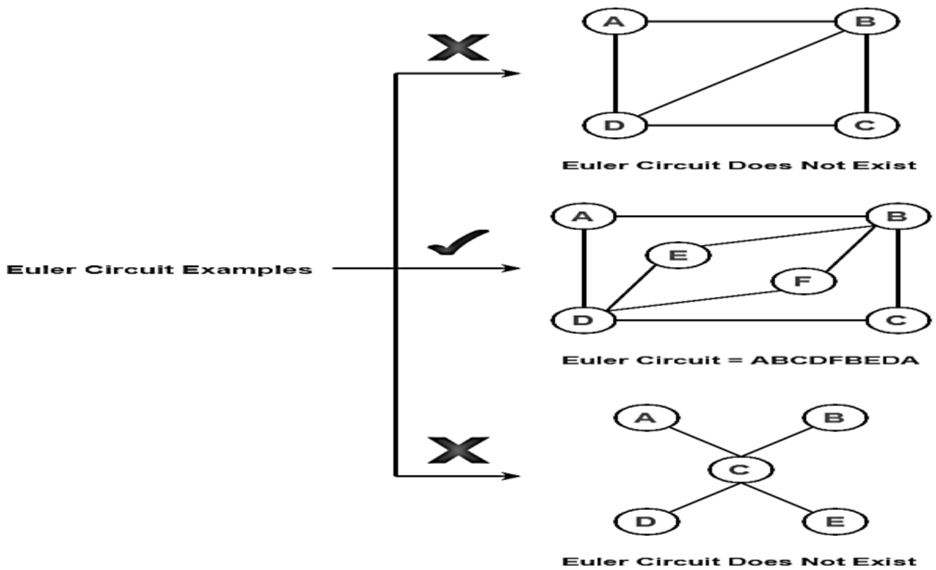
- A closed Euler trail is called as an Euler circuit.

<u>**NOTE**</u>

A graph will contain an Euler circuit if and only if all its vertices are of even degree.

Euler Circuit Examples-

Examples of Euler circuit are as follows



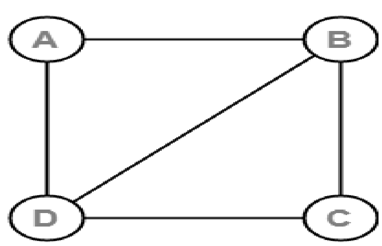
Semi-Euler Graph-

If a connected graph contains an Euler trail but does not contain an Euler circuit, then such a graph is called as a semi-Euler graph.

Thus, for a graph to be a semi-Euler graph, following two conditions must be satisfied-

- Graph must be connected.
- Graph must contain an Euler trail.

Example:



Semi-Euler Graph

Here,

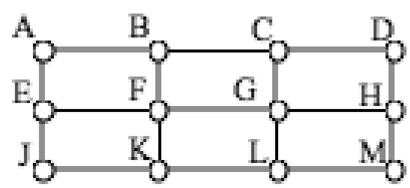
- This graph contains an Euler trail BCDBAD.
- But it does not contain an Euler circuit.
- Therefore, it is a semi-Euler graph.

5.8 Hamiltonian circuit and Hamiltonian path

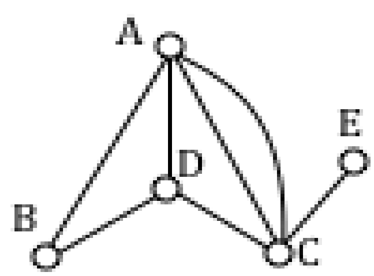
A **Hamiltonian circuit** is a circuit that visits every vertex once with no repeats. Being a circuit, it must start and end at the same vertex.

A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.

EXAMPLE 3
One Hamiltonian circuit is shown on the graph below.
There are several other Hamiltonian circuits possible on this graph.
Notice that the circuit only has to visit every vertex once; it does not need to use every edge.
This circuit could be notated by the sequence of vertices visited, starting and ending at the same vertex: ABFGCDHMLKJEA.
Notice that the same circuit could be written in reverse order, or starting and ending at a different vertex.

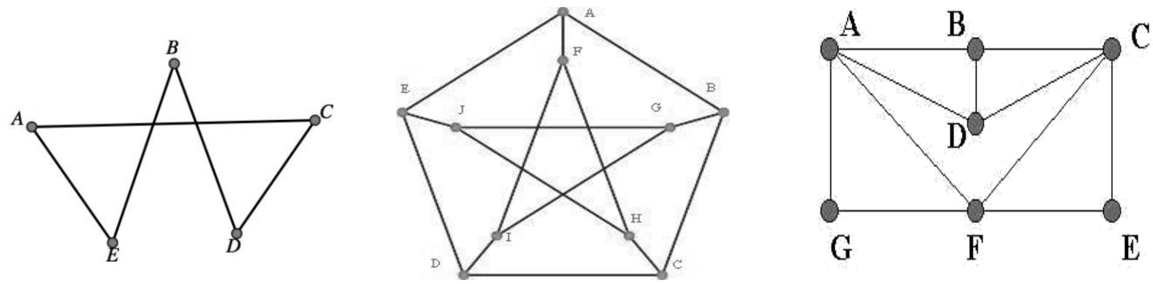


Example 4
Does a Hamiltonian path or circuit exist on the graph below?



We can see that once we travel to vertex E there is no way to leave without returning to C, so there is no possibility of a Hamiltonian circuit.
If we start at vertex E we can find several Hamiltonian paths, such as ECDAB and ECABD

Example 5



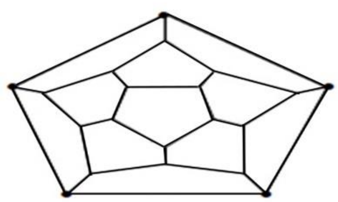
a b c

Graph a. has a Hamilton circuit (one example is ACDBEA)

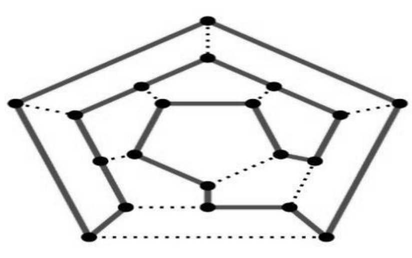
Graph b. has no Hamilton circuits, though it has a Hamilton path (one example is ABCDEJGIFH)

Graph c. has a Hamilton circuit (one example is AGFECDBA)

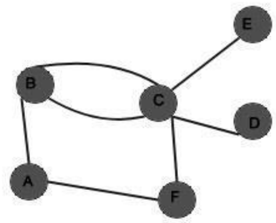
Example 6**-** Does the following graph have a Hamiltonian Circuit?



Solution- Yes, the above graph has a Hamiltonian circuit. The solution is –



Example 7**-** Does the following graph have a Hamiltonian Circuit?



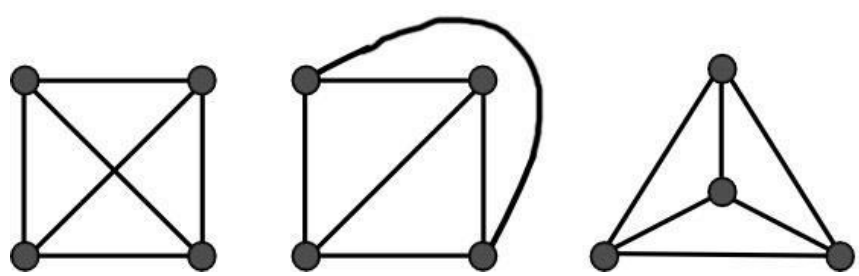
- Solution**-** No the above graph does not have a Hamiltonian circuit as there are two vertices with degree one in the graph.

5.9 Planar graph

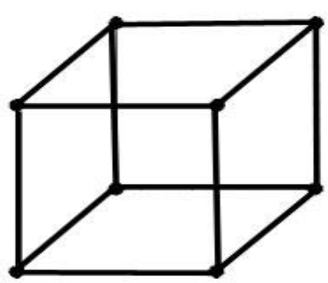
In graph theory, a planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.

Example 8

Consider the complete graph and its two possible planar representations

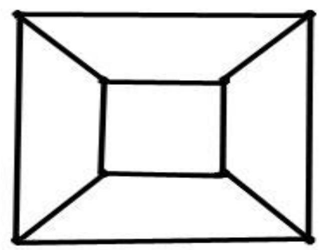


Example 9**–** Is the hypercube planar?



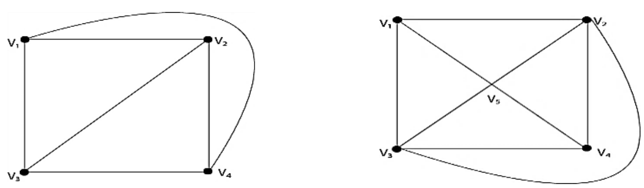
Solution

Yes, planar. Its planar representation



Example 10

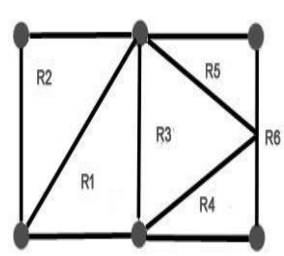
The graph shown in fig is planar graph.



Regions in Planar Graphs

The planar representation of a graph splits the plane into **regions**. These regions are bounded by the edges except for one region that is unbounded.

For example, consider the following graph



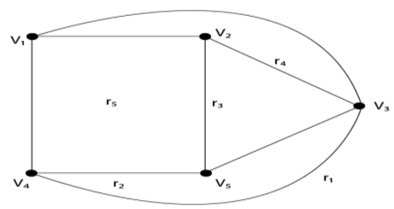
There are a total of 6 regions with 5 bounded regions and 1 unbounded region . All the planar representations of a graph split the plane in the same number of regions. Euler found out the number of regions in a planar graph as a function of the number of vertices and number of edges in the graph

Region of a Graph: Consider a planar graph G=(V,E).A region is defined to be an area of the plane that is bounded by edges and cannot be further subdivided. A planar graph divides the plans into one or more regions. One of these regions will be infinite.

Finite Region: If the area of the region is finite, then that region is called a finite region.

Infinite Region: If the area of the region is infinite, that region is called a infinite region. A planar graph has only one infinite region.

Example 11**:** Consider the graph shown in Fig. Determine the number of regions, finite regions and an infinite region.



Solution: There are five regions in the above graph, i.e. r_1,r_2,r_3,r_4,r_5.

There are four finite regions in the graph, i.e., r_2,r_3,r_4,r_5.

There is only one finite region, i.e., r_1

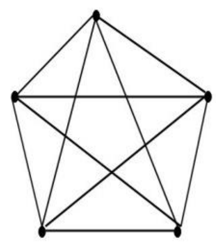
Properties of Planar Graphs:

1. If a connected planar graph G has e edges and r regions, then $r \leq \frac{2}{3}$ e.
2. If a connected planar graph G has e edges, v vertices, and r regions, then v-e+r=2.
3. If a connected planar graph G has e edges and v vertices, then 3v-e≥6.
4. A complete graph K_n is a planar if and only if n<5.
5. A complete bipartite graph K_{mn} is planar if and only if m<3 or n>3.

Non-Planar Graph:

A graph is said to be non planar if it cannot be drawn in a plane so that no edge cross.

Example12**:** The graphs shown in fig are non planar graphs.



K_5

These graphs cannot be drawn in a plane so that no edges cross hence they are non-planar graphs.

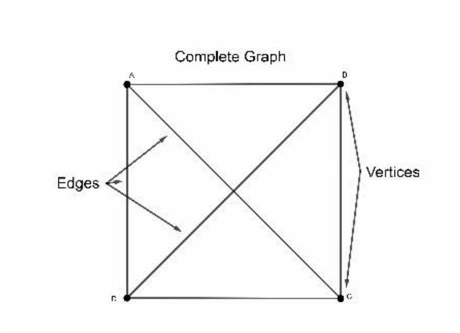
Properties of Non-Planar Graphs

A graph is non-planar if and only if it contains a subgraph homeomorphic to K_5 or $K_{3,3}$

5.10 Complete graph

A **complete graph** is a graph in which each vertex is connected to every other vertex.

That is, a complete graph is an undirected graph where every pair of distinct vertices is connected   Below is an image in Figure showing the different parts of a complete graph:



Characteristics of Complete Graph:

The main characteristics of a complete graph are:

1. **Connectedness:** A complete graph is a connected graph, which means that there exists a path between any two vertices in the graph.

2. **Count of edges:** Every vertex in a complete graph has a degree (n-1), where n is the number of vertices in the graph. So total edges are n*(n-1)/2.
3. **Symmetry:** Every edge in a complete graph is symmetric with each other, meaning that it is un-directed and connects two vertices in the same way.
4. **Transitivity:** A complete graph is a transitive graph, which means that if vertex A is connected to vertex B and vertex B is connected to vertex C, then vertex A is also connected to vertex C.

5. **Regularity:** A complete graph is a regular graph, meaning that every vertex has the same degree.

How to Identify Complete Graph?

To identify a complete graph, you need to check if every vertex in the graph is connected to every other vertex. Here are two methods for identifying a complete graph:

- **Check the degree of each vertex:** In a complete graph with n vertices, every vertex has degree **n-1**. So, if you can determine that every vertex in the graph has degree **n-1**, then the graph is a complete graph.
- **Check the number of edges:** A complete graph with n vertices has **n*(n-1)/2** edges. So, if you can count the number of edges in the graph and verify that it has **n*(n-1)/2** edges, then the graph is a complete graph.

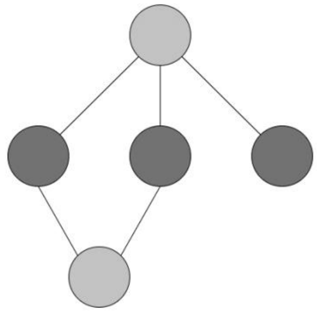
Applications of Complete Graph :

Complete graphs have many applications in various fields, including:

- **Transportation networks:** Complete graphs can be used to represent transportation networks, such as motorways or aircraft routes, where every point is connected to every other location directly.
- **Network analysis:** The characteristics of other graphs, such as clustering, connectedness, or shortest path length, can be measured against the attributes of complete graphs.
- **Optimization:** Complete graphs are used in optimization problems, such as maximum clique, which involves finding the widest subset of vertices in a graph that are all mutually adjacent.

5.11 Bipartite Graph

Formally, a graph G = (V, E) is bipartite if and only if its vertex set V can be partitioned into two non-empty subsets X and Y, such that every edge in E has one endpoint in X and the other endpoint in Y. This partition of vertices is also known as bi-partition.



Example of Bipartite Graph

Characteristics of Bipartite Graph

- Vertices can be divided into two disjoint sets: A bipartite graph can be partitioned into two sets of vertices, with no edges between vertices within each set.
- Every edge connects vertices in different sets: Every edge in a bipartite graph connects a vertex from one set to a vertex from the other set.
- No odd-length cycles: A bipartite graph cannot contain any odd-length cycles, as this would require vertices from the same set to be connected by an edge.
- Maximum degree is bounded by the size of the smaller set: The maximum degree of a vertex in a bipartite graph is equal to the size of the smaller set.
- Coloring with two colors: A bipartite graph can be colored with two colors,, such that no adjacent vertices have the same color.

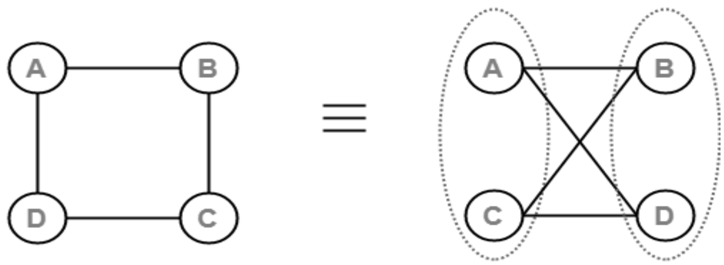
Application of Bipartite Graph

Bipartite graphs have many applications in different fields, including:

- **Matching problems:** Bipartite graphs are commonly used to model matching problems, such as matching job seekers with job vacancies or assigning students to project supervisors. The bipartite structure allows for a natural way to match vertices from one set to vertices in the other set.
- **Social networks:** Bipartite graphs, where the nodes in one set represent users and the nodes in the other set reflect interests, groups, or communities, can be used to simulate social networks. The bipartite form makes it simple to analyse the connections between users and interests.
- **Web Search engine:** The query and click-through data of a search engine can be defined using a bipartite graph, where the two sets of vertices represent queries and web pages.

Example 13

The following graph is an example of a bipartite graph



Example of Bipartite Graph

Here,

The vertices of the graph can be decomposed into two sets.

The two sets are X = {A, C} and Y = {B, D}.

The vertices of set X join only with the vertices of set Y and vice-versa.

The vertices within the same set do not join.

Therefore, it is a bipartite graph.

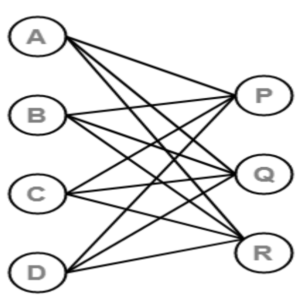
Complete Bipartite Graph

A complete bipartite graph may be defined as follows-

A bipartite graph where every vertex of set X is joined to every vertex of set Y is called as complete bipartite graph OR Complete bipartite graph is a bipartite graph which is complete OR Complete bipartite graph is a graph which is bipartite as well as complete

Example 14

The following graph is an example of a complete bipartite graph-



Example of Complete Bipartite Graph

- The graph is a bipartite graph as well as a complete graph.
- Therefore, it is a complete bipartite graph.
- This graph is called as $\mathbf{K_{4,3}}$.

Bipartite Graph Properties-

Few important properties of bipartite graph are-

- Bipartite graphs are 2-colorable.
- Bipartite graphs contain no odd cycles.
- Every sub graph of a bipartite graph is itself bipartite.
- There does not exist a perfect matching for a bipartite graph with bipartition X and Y if |X| ≠ |Y|.
- In any bipartite graph with bipartition X and Y,

Sum of degree of vertices of set X = Sum of degree of vertices of set Y

5.12 Hypercube graph

A hypercube is an nn-dimensional analogue of a square (nn = 2) and a cube (nn = 3). It has $2n2n$ vertices and $2n−1n2n−1n$ edges.

The hypercube graph Qn

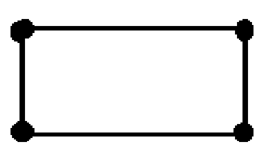
Qn is the graph formed from the vertices and edges of an nn-dimensional hypercube with nn edges touching each vertex. For instance, the hypercube graph $Q4Q4$ is the graph formed by the 16 vertices and 32 edges of a 4-dimensional hypercube.

Q(0) with vertex 1

(Single vertex)

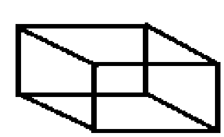
Q(1) with vertex 2

(Complete graph)



Q(2) with vertex 4

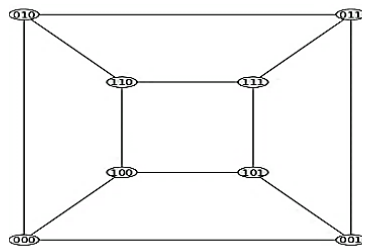
(Cycle of length 4)



Q(3) with vertex 8

(Cubical graph)

Example 15



5.13 Matrix representations of graphs

Matrix representations of graphs go back a long time and are still in some areas the only way to represent graphs.

Adjacency matrices represent adjacent vertices and incidence matrix vertex-edge incidences. Both are fully capable of representing undirected and directed graphs.

Adjacency Matrix

An Adjacency Matrix A[V][V] is a 2D array of size V × V where VV is the number of vertices in a undirected graph. If there is an edge between V_x to V_y then the value of $A[V_x][V_y]=1$ and $A[V_y][V_x]=1$, otherwise the value will be zero.

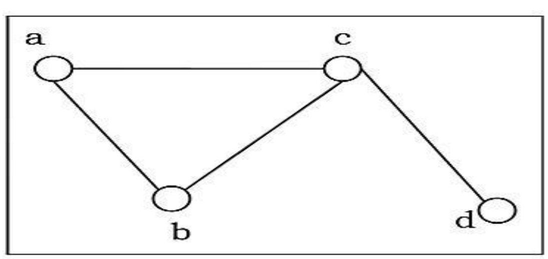
For a directed graph, if there is an edge between V_x to V_y, then the value of $A[V_x][V_y]=1$, otherwise the value will be zero.

Properties of Adjacency matrix

1. It is symmetric in non-directed graph and not always symmetric in directed graph.
2. In the absence of loops, entries in the principal diagonal are all zero.

Adjacency Matrix of an Undirected Graph

Let us consider the following undirected graph and construct the adjacency matrix −

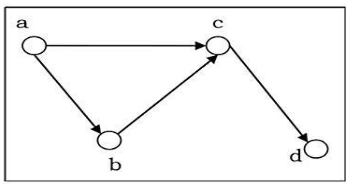


Adjacency matrix of the above undirected graph will be

	a	b	c	d
a	0	1	1	0
b	1	0	1	0
c	1	1	0	1
d	0	0	1	0

Adjacency matrix of directed graph

Let us consider the following directed graph and construct its adjacency matrix



Adjacency matrix of the above directed graph will be

	a	b	c	d
a	0	1	1	0
b	0	0	1	0
c	0	0	0	1
d	0	0	0	0

Incidence Matrix

The *incidence matrix A* of an *undirected* graph has a row for each vertex and a column for each edge of the graph.

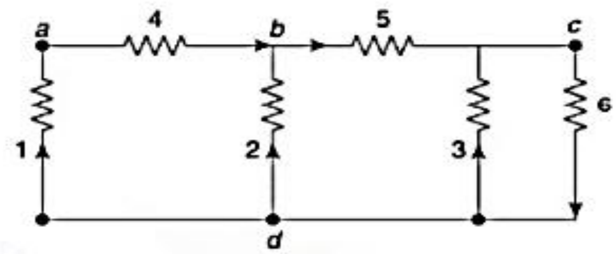
The element $A_{[[i,j]]}$ of A is 1 if the i^{th} vertex is a vertex of the j^{th} edge and 0 otherwise.

The *incidence matrix A* of a *directed* graph has a row for each vertex and a column for each edge of the graph.

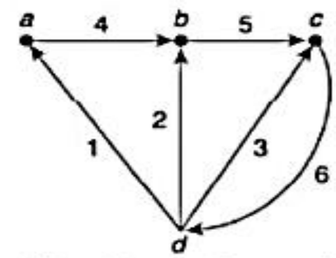
The element $A_{[[i,j]}$ of A is − 1 if the i^{th} vertex is an initial vertex of the j^{th} edge, 1 if the i^{th} vertex is a terminal vertex, and 0 otherwise.

Example 16

Consider a network having 6 branches and 4 nodes as shown below. Node d is taken as a reference node. The equivalent graph of the network is also drawn as shown below. The order of the incidence matrix will be 4×6.



(a) *Network*



(b) *Graph of network*

The incidence matrix $[A]_{ij}$ can be written as,

		Branches					
		1	2	3	4	5	6
Nodes	*a*	−1	0	0	+1	0	0
	b	0	−1	0	−1	+1	0
	c	0	0	−1	0	−1	+1
Reference Node	*d*	+1	+1	+1	0	0	−1

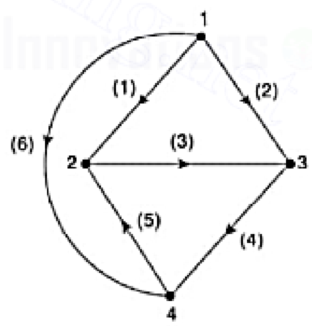
Properties of Incidence Matrix

A complete incidence matrix $[A]_{ij}$ a graph has some properties by which one can identify whether the given matrix is a complete incidence matrix or not. These properties are:

- The sum of values of $[A]_{ij}$ of any column is equal to zero.
- For a closed loop system, the determinant of $[A]_{ij}$ is always zero.
- The rank of the complete incidence matrix $[A]_{ij}$ is n-1 where n=number of nodes in the graph.

Example 17

Find the possible number of trees for the graph shown below.



Solution: For the graph, let the node 4 be the reference node. The complete incidence matrix can be written as,

$$[A]_{ij} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix}$$

From the complete incidence matrix, the reduced incidence matrix can be obtained as,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

Number of trees can be calculated as,

$$N = \left| [A_r][A_r]^T \right|$$

$$= \det\left\{ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\} = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 16$$

Self-assessment questions

1.Explain the types of graphs

2.Write in detail about connectivity

3.Illustrate Euler graph with example

4.Elucidate Hamilton circuits and path

5.Represent the adjacency of matrix with example

6.Discuss planar graph

7.Analyse Hypercube graph

8.Eloborate complete graph

Let us sum up

Graph theory is a branch of mathematics focused on the study of graphs, which are structures consisting of vertices (or nodes) connected by edges (or links). In graph theory, a graph can be either undirected, where edges have no direction, or directed, where edges have a specified direction from one vertex to another. Graphs can also be weighted, with edges having associated costs or values, or unweighted, where edges are considered equal. Special types of graphs include complete graphs, where every pair of vertices is connected, and trees, which are connected, acyclic graphs. Properties of graphs such as paths and cycles are central, with connected graphs having paths between all pairs of vertices and planar graphs being drawable on a plane without edge crossings. Graph theory includes important algorithms for practical problems, such as finding the shortest path between nodes using Dijkstra's or Bellman-Ford algorithms, traversing graphs with Breadth-First Search (BFS) or Depth-First Search (DFS), and constructing Minimum Spanning Trees using Kruskal's or Prim's algorithms. Applications of graph theory are diverse, ranging from network analysis and optimization to solving scheduling problems and modeling biological networks.

Check your progress

1. In a 7-node directed cyclic graph, the number of Hamiltonian cycle is to be ______

a) 728
b) 450
c) 360
d) 260

2. If each and every vertex in G has degree at most 23 then G can have a vertex colouring of __________

a) 24
b) 23
c) 176
d) 54

3. Triangle free graphs have the property of clique number is __________
a) less than 2
b) equal to 2
c) greater than 3
d) more than 10

4. Berge graph is similar to ______ due to strong perfect graph theorem.
a) line graph
b) perfect graph
c) bar graph
d) triangle free graph

5. Let D be a simple graph on 10 vertices such that there is a vertex of degree 1, a vertex of degree 2, a vertex of degree 3, a vertex of degree 4, a vertex of degree 5, a vertex of degree 6, a vertex of degree 7, a vertex of degree 8 and a vertex of degree 9. What can be the degree of the last vertex?
a) 4
b) 0
c) 2
d) 5

6. A ______ is a graph which has the same number of edges as its complement must have number of vertices congruent to 4m or 4m modulo 4(for integral values of number of edges).
a) Subgraph
b) Hamiltonian graph
c) Euler graph
d) Self complementary graph

7. In a ______ the vertex set and the edge set are finite sets.
a) finite graph
b) bipartite graph
c) infinite graph
d) connected graph

8. If G is the forest with 54 vertices and 17 connected components, G has _______ total number of edges.
a) 38
b) 37

c) 17/54
d) 17/53

9. The number of edges in a regular graph of degree 46 and 8 vertices is ____________

a) 347
b) 230
c) 184
d) 186

10. An undirected graph G has bit strings of length 100 in its vertices and there is an edge between vertex u and vertex v if and only if u and v differ in exactly one bit position. Determine the ratio of the chromatic number of G to the diameter of G?
a) $1/2^{101}$
b) 1/50
c) 1/100
d) 1/20

Unit summary

Graph theory studies graphs, which consist of vertices (nodes) connected by edges (links). Graphs can be undirected or directed, and weighted or unweighted. Key concepts include paths (sequences of connected edges), cycles (paths that return to the starting point), and trees (connected, acyclic graphs). Essential algorithms address finding shortest paths, traversing graphs, and constructing Minimum Spanning Trees (MST). Applications range from network optimization to scheduling and **biological modeling.**

Glossary

Graph: A collection of vertices (nodes) connected by edges (links).

Vertex (Node): A fundamental unit or point in a graph.

Edge (Link): A connection between two vertices in a graph.

Undirected Graph: A graph where edges have no direction; if an edge connects vertex A to vertex B, it also connects B to A.

Directed Graph (Digraph): A graph where edges have a direction; an edge from vertex A to vertex B does not imply an edge from B to A.

Weighted Graph: A graph in which edges have weights or costs associated with them.

Unweighted Graph: A graph where edges do not have weights; all edges are considered equal.

Path: A sequence of edges connecting a sequence of vertices.

Cycle: A path that starts and ends at the same vertex without repeating edges or vertices.

Connected Graph: A graph where there is a path between every pair of vertices.

Disconnected Graph: A graph where some pairs of vertices are not connected by a path.

Tree: A connected, acyclic undirected graph.

Complete Graph: A graph where every pair of distinct vertices is connected by a unique edge.

Bipartite Graph: A graph where vertices can be divided into two disjoint sets such that no two vertices within the same set are adjacent.

Subgraph: A graph formed from a subset of another graph's vertices and edges.

Planar Graph: A graph that can be drawn on a plane without any edges crossing.

Eulerian Path: A path that visits every edge of a graph exactly once.

Eulerian Circuit: An Eulerian Path that starts and ends at the same vertex.

Hamiltonian Path: A path that visits every vertex exactly once.

Hamiltonian Circuit: A Hamiltonian Path that starts and ends at the same vertex.

Suggested readings

Graph Theory: A Problem Oriented Approach by Daniel A. Marcus